# Quasi-treeable equivalence relations

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CBERs are well-studied object. for a survey, see "Countable Borel Equivalence Relations" by Jackson-Kechris-Louveau.

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**Theorem [Feldman-Moore, '77]**

All countable Borel equivalence relations arise as orbit equivalence relations.

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### Reductions

If $(X, E), (Y, F)$ are two CBERs, a Borel function $f : X \to Y$ such that

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is called a **reduction**. We write $E \leq F$.

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**smooth** $<$ **hyperfinite** $<$ **treeable** $<$ **(non-treeable)**

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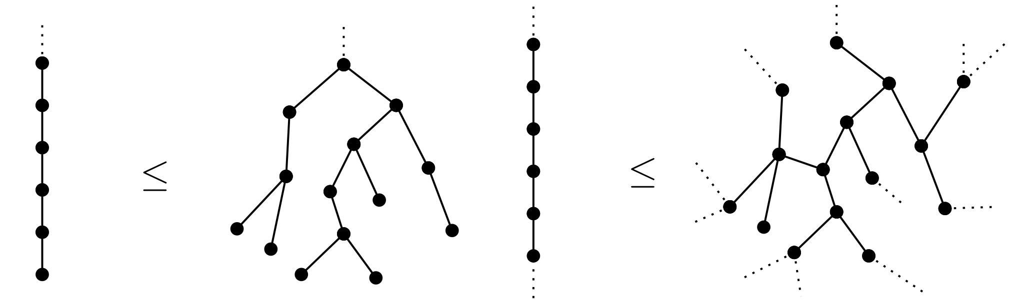
We often require the graphings to satisfy extra conditions.

Many of these conditions give measure of complexity: if $E \leq F$ and $F$ can be given a treeing, (in other words, is treeable), then $E$ is also treeable.



**SMOOTH**
**Natural line**

**HYPERFINITE**
**One or two ended trees**

**TREEABILITY**
**Arbitrary trees**

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**Theorem (Classical)**

*A f.g. group $\Gamma$ is virtually free iff it has a l.f Cayley graph $G$ which is a* ***quasi-tree****.*

**Quasi-tree** $\leftarrow$ graph quasi-isometric to a tree $\exists f : G \to T$ which

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There are $M > 1, K > 0$ s.t.

$$\frac{1}{M} d_T(f(x), f(y)) - K \leq d_G(x, y) \leq M d_T(f(x), f(y)) + K,$$
$$d_T(\mathrm{im}(f), z) \leq K.$$

for all $x, y \in V(G)$ and $z \in V(T)$.

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**Better Question**

If a CBER is **l.f.** quasi-treeable, must it be treeable?

# Results

**Theorem (R. Chen, A. P., R. Tao, A. Tserunyan 2023+)**

*Let $E \subseteq X^2$ be a CBER, $G \subseteq E$ be a locally finite graphing whose each component is a quasi-tree.*

(i) *$G$ is treeable.*

(ii) *If $G$ is one-ended, then $E$ is hyperfinite.*

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**Theorem (R. Chen, A. P., R. Tao, A. Tserunyan 2023+)**

*Let $E \subseteq X^2$ be a CBER, $G \subseteq E$ be a locally finite graphing whose each component is a quasi-tree. If $G$ has a global bound on degree, there is a reduction to a Borel tree $(Y, \mathcal{T})$ which is a quasi-isometry (class-wise).*

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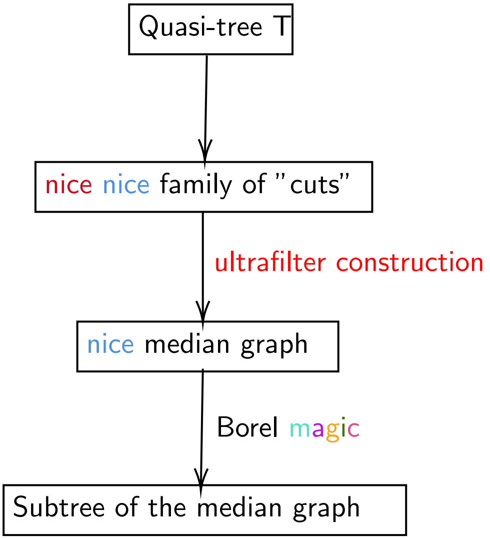
For the rest of the talk, $T$ is a locally finite connected quasi-tree.

# Overview of the proof



# Cuts

Given a countable graph $G$, the set of cuts of $G$ is

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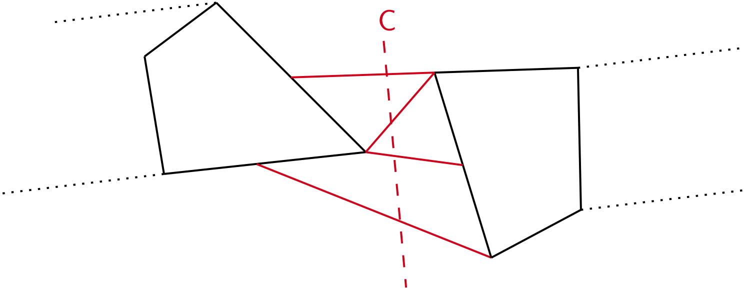

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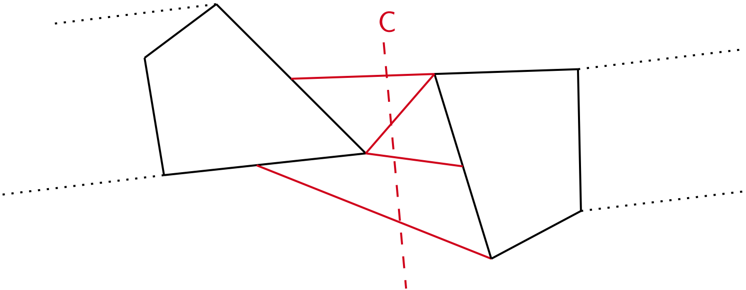


The two components of $G - C$ are called the **sides** of $C$.

# Cuts in Quasi-tree

Recall that $T$ is a quasi-tree. Given $R \in \mathbf{N}$, look cuts of bounded diameter:

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# Pocset of sides

For the right $R$, as per the last slide, the collection

$$\mathcal{P}_R(T) = \{P \subset V(T) : P \text{ is a side of some } C \in \mathcal{C}_R(T)\} \cup \{\varnothing, V(T)\}$$

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**Theorem [Isbel '80 + Werner '81]**

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$$\{\text{"}nice\text{" } pocsets\ \mathcal{P}\} \cong \{median\ graphs\ \mathcal{O}\}$$

We now have a median graph $\mathcal{O}_R(T)$. The last step is to find a subtree and then we are done. But what is a median graph?

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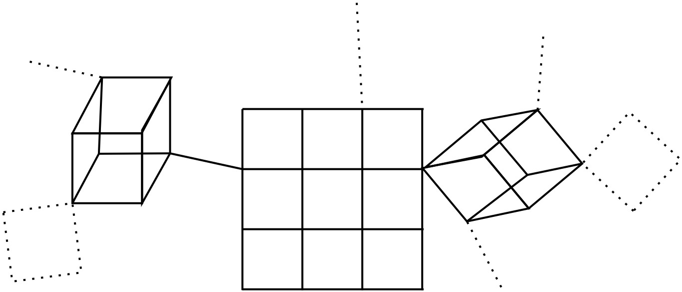

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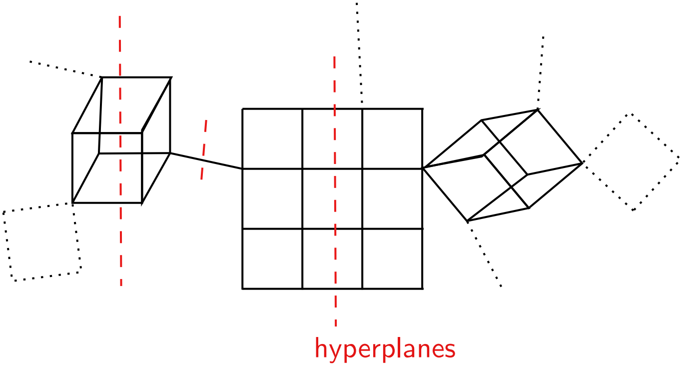
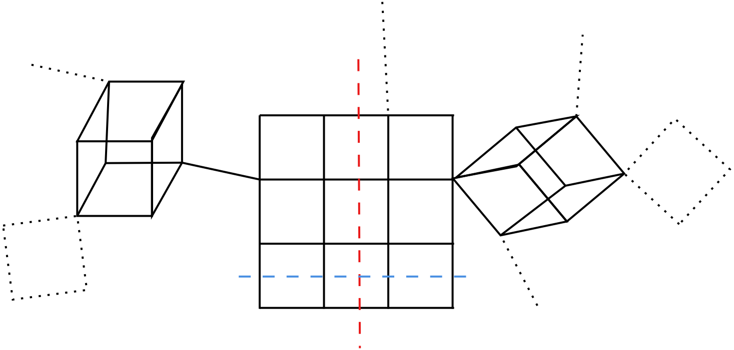


# Perpendicular hyperplanes



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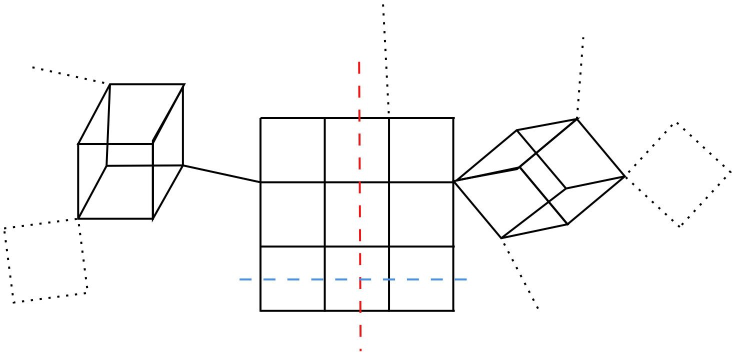

perpendicular hyperplanes

Hyperplanes are **perpendicular** if all pair of sides intersect.

# Revisiting Isbel-Werner duality

| $\mathcal{C}_R(T)$ | Median graph $\mathcal{O}_R(T)$ |
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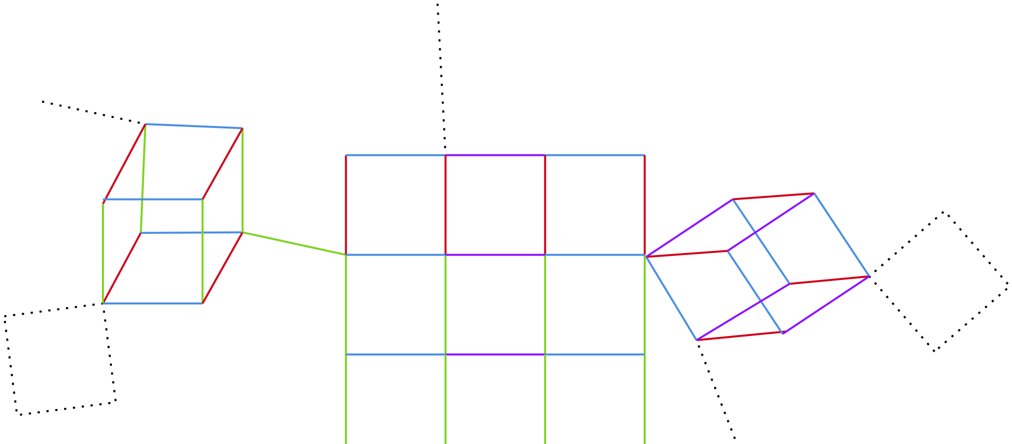
**Theorem (Follows from Kechris-Miller '04)**

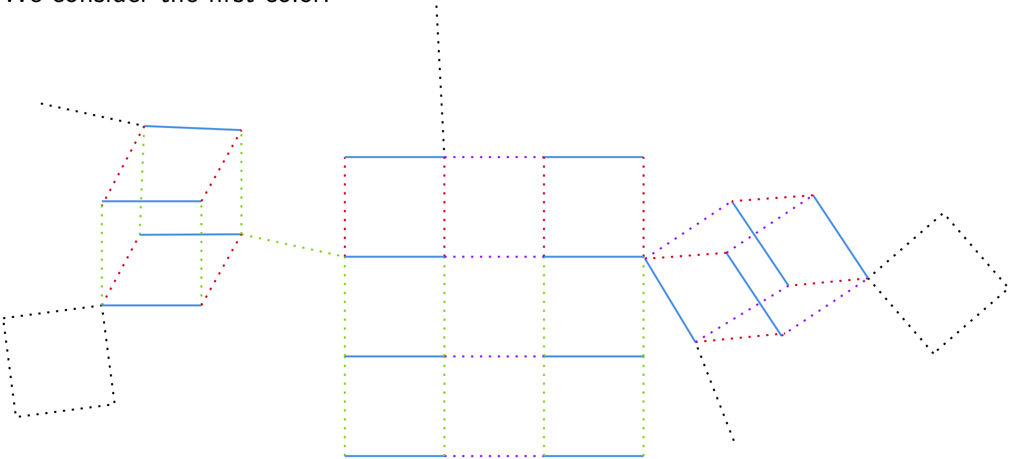
There exists a countable coloring of hyperplanes such that if two hyperplanes are perpendicular, they have different color.

We have a coloring now:



# Building the tree: first color

We consider the first color:



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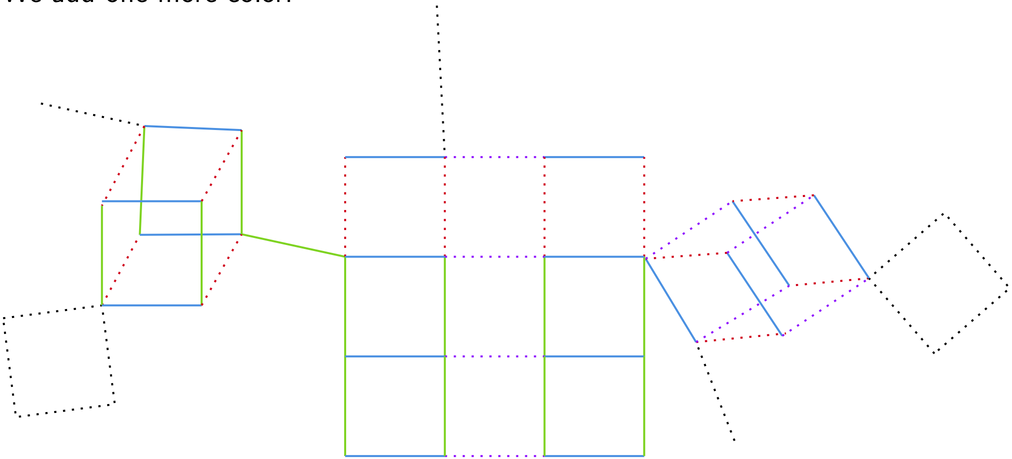

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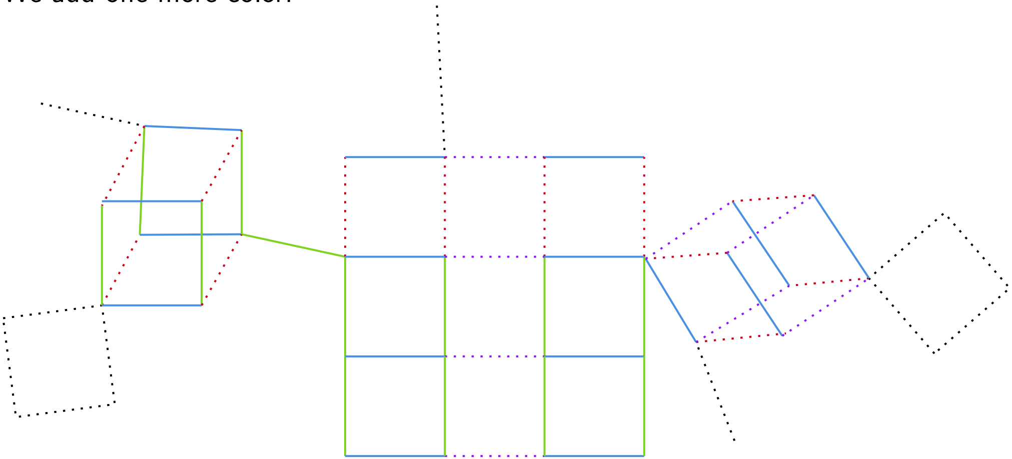

But we don't have a tree anymore!

# Cycle cutting

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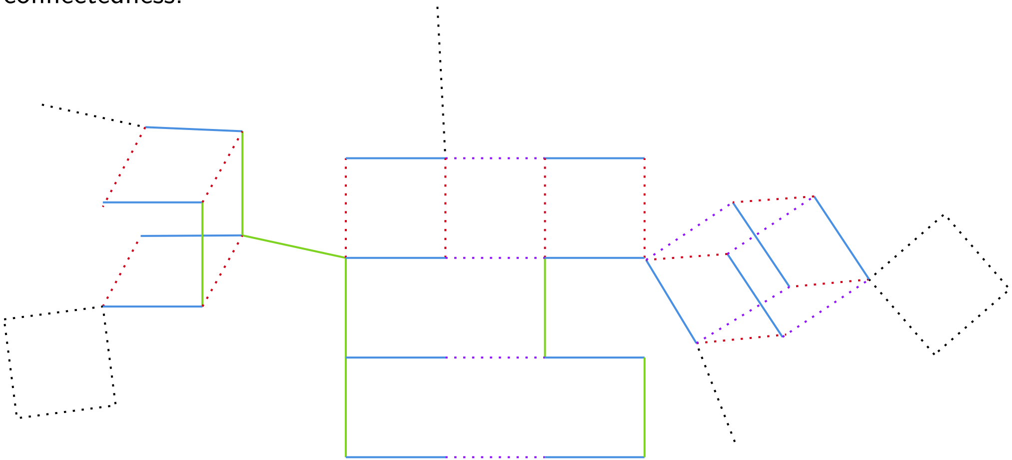

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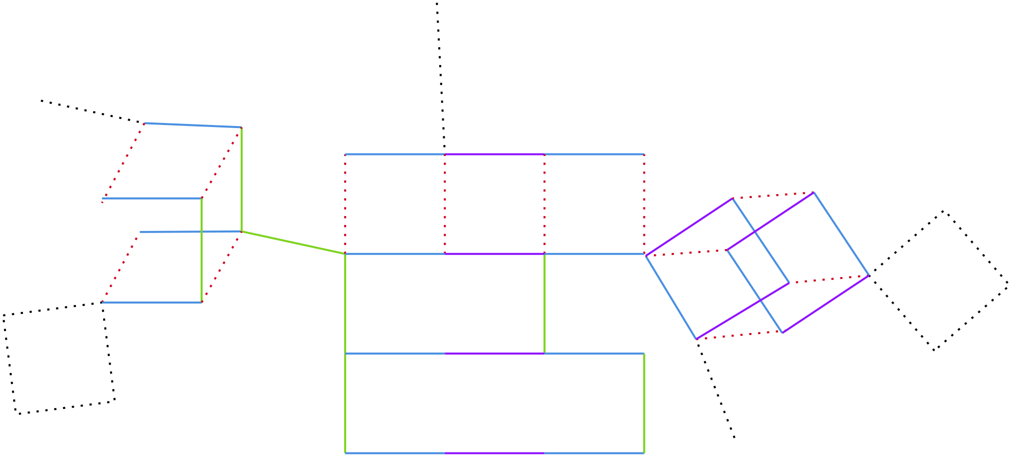

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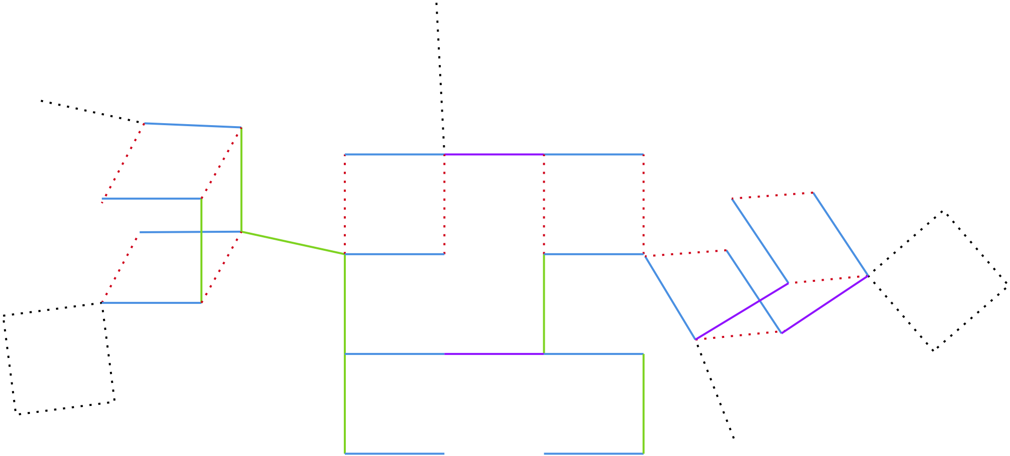

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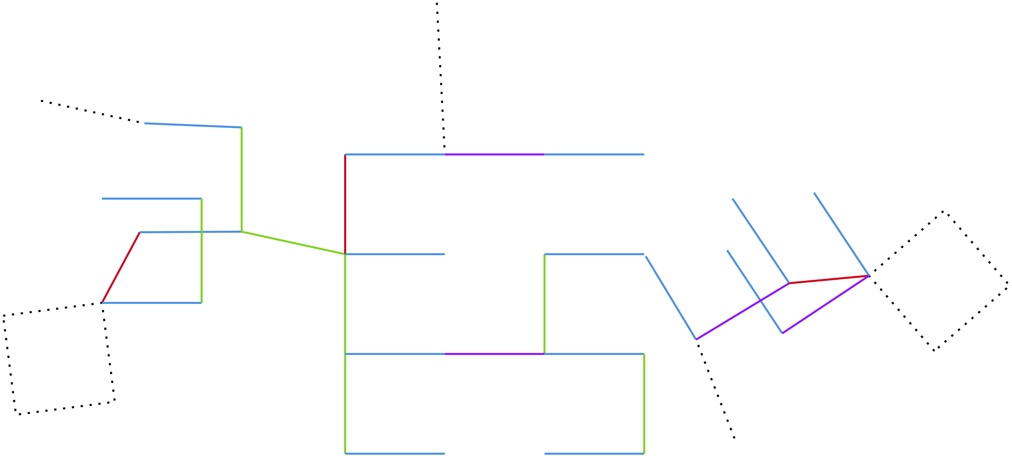

# After all is said and done

**Theorem (R. Chen, A. P., R. Tao, A. Tserunyan 2023+)**

*If $\mathcal{O}$ is a median graph with a countable coloring of hyperplanes such that perpendicular hyperplanes have different colors, there is a "canonical" subtree $\widehat{T} \subset \mathcal{O}$.*

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Can be generalized to other "tree-like" notions for graph:

**Theorem (R. Chen, A. P., R. Tao, A. Tserunyan 2023+)**

*If a CBER $E$ admits a locally finite graphing with components quasi-trees or of bounded tree-width, then $E$ is treeable.*

# After all is said and done

**Theorem (R. Chen, A. P., R. Tao, A. Tserunyan 2023+)**

*If $\mathcal{O}$ is a median graph with a countable coloring of hyperplanes such that perpendicular hyperplanes have different colors, there is a "canonical" subtree $\widehat{T} \subset \mathcal{O}$.*

Can be generalized to other "tree-like" notions for graph:

**Theorem (R. Chen, A. P., R. Tao, A. Tserunyan 2023+)**

*If a CBER $E$ admits a locally finite graphing with components quasi-trees or of bounded tree-width, then $E$ is treeable.*

# Thank you!