

Measure equivalence of Baumslag-Solitar groups & type III relations

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Definition (Baumslag-Solitar groups)

Let $p, q \in \mathbf{Z} - \{0\}$. The **Baumslag-Solitar group** with parameters (p, q) is:

$$\text{BS}(p, q) = \langle a, t \mid ta^p t^{-1} = a^q \rangle$$

Introduced in '62 by G. Baumslag et D. Solitar. First example of a 2-gen. 1-rel. non-Hopfian group.

[These groups] destroys several conjectures long and strongly held by [me] and others, and some published and unpublished "theorems".
 –B.H. Neumann

- $BS(p, q)$ isom $BS(q, p)$
- $BS(p, q)$ isom $BS(-p, -q)$
- Moldavanskii '91: These are all the isomorphisms.
- $BS(p, q)$ comm $BS(-p, q)$, i.e they have isomorphic finite index subgroup.

Restrict to the case $1 \leq p \leq q$.

Three families of Baumslag-Solitar groups:

- $BS(1, p)$ is solvable, amenable.
- $BS(p, p)$ contains finite index $\mathbf{F} \times \mathbf{Z}$, where \mathbf{F} is f.g free group.
- If $1 < p < q$. More heterogenous class.

Let Γ, Δ be countable groups.

Definition (Measure equivalence)

Γ and Δ are **measure equivalent**, denoted $\Gamma \text{ ME } \Delta$, if there exists actions $\Gamma, \Delta \curvearrowright (\Omega, \mu)$ on a standard measure space (e.g $\mathbf{N}, \mathbf{R}, 2^{\mathbf{N}}$... with Borel measures) such that:

- Actions are pmp, free and commute.
- Both actions admit (standard) fundamental domains of finite measure:

$$\mu(\Gamma \backslash \Omega), \mu(\Delta \backslash \Omega) < \infty$$

Introduced by Gromov in '93 as measured analogue of quasi-isometry. Examples:

- If $\Gamma < \Delta$ finite index subgroup, then $\Gamma \text{ ME } \Delta$.
- If $\Gamma \text{ comm } \Delta$, then $\Gamma \text{ ME } \Delta$.
- If Γ, Δ are both lattices in the same lcsc group G , then $\Gamma \text{ ME } \Delta$.

Measure equivalence, restricted to the family of Baumslag-Solitar groups:

- Ornstein-Weiss '80 implies $BS(1, p)$ forms a ME-class.
- Commensurability implies all $BS(p, p)$ are ME.
- Kida '14 proves $BS(p, p)$ actually form a ME-class.
- Partial results for the third family.

Theorem (D. Gaboriau, P., A. Tserunyan, R. Tucker-Drob, K. Wróbel, '24+)

For $1 < p < q$ and $1 < r < s$, we have the measure equivalence:

$$\text{BS}(p, q) \text{ ME } \text{BS}(r, s).$$

Completes the classification of $\text{BS}(p, q)$ up to ME.

(X, μ) standard probability space e.g. $[0, 1]$ with Lebesgue measure.

$E \subset X^2$ Borel equivalence relation with countable classes.

E preserves the measure class (**pcm**) of μ : if A is null, the saturation $[A]_E$ is null.

All such equivalence relations arises as non-singular action of countable groups (Feldman – Moore '77)

(X, μ, E) pcm: Radon-Nikodym cocycle $\mathfrak{w} : E \rightarrow (0, \infty)$ satisfying:

- For almost all $x \in X$, for all $y, z \in [x]_E$:
 $\mathfrak{w}(z, y)\mathfrak{w}(y, x) = \mathfrak{w}(z, x)$
- For all $f : E \rightarrow [0, \infty]$:

$$\int_X \sum_{y \in [x]_E} f(x, y) \, d\mu = \int_X \sum_{y \in [x]_E} f(y, x)\mathfrak{w}(y, x) \, d\mu$$

Denote $T_{p,q}$ for the transitive directed tree with $\text{indeg} \equiv q$ and $\text{outdeg} \equiv p$

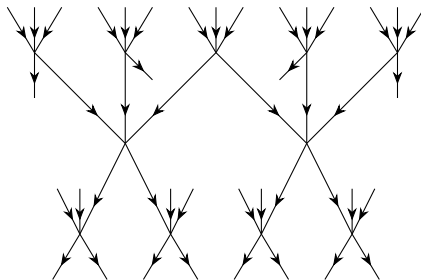


Figure: $T_{2,3}$

Definition (Cocycle $T_{p,q}$ treeing)

A **cocycle $T_{p,q}$ treeing** on (X, μ) is a Borel directed graph $G \subset X^2$ such that:

- Each connected component is isomorphic to $T_{p,q}$
- For each directed edge $y \rightarrow x$,

$$\mathfrak{w}(y, x) = \frac{p}{q}$$

Abbreviate to **CT** $_{p,q}$.

$CT_{p,q}$ on $(X, \mu) \iff$ free pmp $\text{Aut}(T_{p,q}) \curvearrowright (\tilde{X}, \tilde{\mu})$

$\text{Aut}(T_{p,q}) \text{ ME } \text{Aut}(T_{r,s}) \implies \text{BS}(p, q) \text{ ME } \text{BS}(r, s)$ (not easy)

Question

Given (X, ν, G) a $CT_{r,s}$, does there exist a prob. measure $\mu \sim \nu$ and a graph G' with the same conn. comp. s.t (X, μ, G') is a $CT_{p,q}$?

Cost and related notions give obstructions to similar questions.

Crucial definition:

Definition

(X, μ, E) pcm is **type III** if there are no σ -finite measures $\nu \sim \mu$ preserved by E , aka $\mathfrak{m}_\mu \equiv 1$.

All $CT_{p,q}$ are type III.

Equivalently, if all complete sections $A, B \subset X$ are in bijection with a full pseudo-group element.

Theorem (P., '24+)

If (X, μ, E) is pcm, type III, aperiodic, treeable and $d \in \{2, \dots, \infty\}$, then there is a free mcp action $\mathbf{F}_d \curvearrowright (X, \mu)$ such that $E = E_{\mathbf{F}_d}$ and the Schreier graph has no vanishing ends.

Without vanishing ends \rightarrow True in a Borel way for compressible CBER. False for II_∞ .

Techniques to prove this theorem are useful to construct $CT_{p,q}$.

Let (X, E, μ) be pcm.

Definition

The **Maharam extension** of E on $X \times \mathbf{R}$ is defined by

$$(x, r) E^M (y, s) \iff x E y \ \& \ r - \log \mathfrak{m}(y, x) = s$$

The Maharam E^M preserves the measure $\mu \times e^t dt$.

The flow $\mathbf{R} \curvearrowright X \times \mathbf{R}$ on the second coordinate preserves E^M .

Let (X, E, μ) be pcm and E^M its Maharam extension on $X \times \mathbf{R}$.

Definition

The **Krieger flow** \mathbf{K}_E is the induced mcp flow $\mathbf{R} \curvearrowright \Omega$ on the space of ergodic components of E^M . [▶ More info.](#)

For our proof, Kronecker flows on torii $\mathbf{R} \curvearrowright [0, \log \frac{s}{r}) \times [0, \log \frac{q}{p})$.

Krieger flow of a pmp action of $\text{Aut}(T_{r,s}) \curvearrowright U \times [0, \log \frac{q}{p})$, where U is mixing and act by modular homomorphism on second coordinate.

If $F \leq E$ is a subequivalence relation, there is mcp factor map $\mathbf{K}_F \rightarrow \mathbf{K}_E$.

Theorem (D. Gaboriau, P., A. Tserunyan, R. Tucker-Drob, K. Wróbel, '24+)

Let (X, μ, G) be a type III mcp graph with aperiodic Krieger flow \mathbf{K}_E . Then there is a hyperfinite subgraph $H \leq G$ for which the factor $\mathbf{K}_H \rightarrow \mathbf{K}_G$ is a conjugacy.

Question: What about periodic Krieger flow?

Thank you!

Extra info : Space of ergodic components

Let (X, E, μ) be a pcm CBER.

An ergodic decomposition is a E -invariant map $\pi : X \rightarrow Z$ with the universal property: every E -invariant map $\rho : X \rightarrow Y$ factors through π : there is $f : Z \rightarrow Y$ such that $\rho = f \circ \pi$.

▶ return