Borel Complexity of Archimedean Orders

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Throughout, Γ is a countable group.

Definition (Left-Orders on Groups)

A total order $<$ on Γ is a **left-order** if it is invariant by multiplication on the left:

$$h < k \iff gh < gk$$

An equivalent formulation is the following:

Definition (Positive Cones)

A subset $P \subset \Gamma$ is a **positive cone** if:

- P is a semigroup: $P \cdot P \subset P$.
- The group Γ is partitioned as $\Gamma = P \sqcup \{1\} \sqcup P^{-1}$.

Given a left-order $<$, an element g is **positive** if

$$g > 1.$$

Definition (Space of left-orders)

The **space of left-orders** is

$$LO(\Gamma) = \{P \subset \Gamma : P \text{ is a positive cone.}\}$$

This is a compact Polish space. There is a Borel action $Aut(\Gamma) \curvearrowright LO(\Gamma)$ given by

$$h\,(\phi \cdot <)\,k \quad \longleftrightarrow \quad \phi^{-1}(h) < \phi^{-1}(k)$$

This is the **automorphism action**. It also restricts to the **conjugacy action** $\Gamma \curvearrowright LO(\Gamma)$.

Definition (Archimedean Order + Space)

A left-order is **Archimedean** if for all $g, h > 1$, there exists $n \in \mathbb{N}$ such that

$$g < h^n.$$

The space of Archimedean orders $Ar(\Gamma)$ is a G_δ subset of $LO(\Gamma)$.

Theorem (Hölder)

If $(\Gamma, <)$ *is an Archimedean order**ed** group, it is isomorphic (as ordered groups) to a subgroup of* $\mathbb{R}$*. In particular, the only f.g Archimedean orderable groups are isomorphic (as groups) to* $\mathbb{Z}^n$*.*

Main goal: link properties of Γ to complexity results on $LO(\Gamma)$ or $Ar(\Gamma)$.

Theorem (Calderoni - Clay, 2020)

If $E(\Gamma \curvearrowright LO(\Gamma)$ is smooth, then Γ is locally indicable. (Every f.g subgroups surjects onto $\mathbb{Z}$)

More work has be done by Filippo and Adam studying links to the L-space conjecture.

Theorem (Calderoni - Clay, 2023)

The conjugacy relation for $\Gamma = BS(1,2)$ is bireducible to E_0, providing the first such example.

Theorem (Calderoni - Marker - Motto Ros - Shani)

The equivalence relation $E(GL_2(\mathbb{Q}) \curvearrowright Ar(\mathbb{Q}^2))$ is not smooth.

Question - CMMRS

- Is $E(GL_2(\mathbb{Q}) \curvearrowright Ar(\mathbb{Q}^2))$ hyperfinite?
- What is the complexity of $E(GL_n(\mathbb{Q}) \curvearrowright Ar(\mathbb{Q}^n))$ for $n \geq 3$?

Theorem (P.)

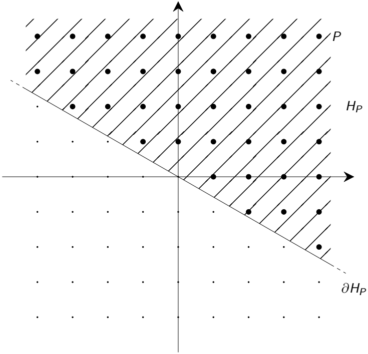
The equivalence relation $E(GL_n(\mathbb{Z}) \curvearrowright Ar(\mathbb{Z}^n))$ is

- *hyperfinite if $n = 2$.*
- *not hyperfinite if $n \geq 3$.*
- *not treeable if $n \geq 4$.*

Let us look at hyperfiniteness. Implicitly, we will often be jumping between index two equivalence relations, corresponding to:

- $SL_n(\mathbb{Z})$ vs $GL_n(\mathbb{Z})$
- Choice of orientation of order: To every order $<$, there is a reverse order $>$.

Hölder's theorem is really about linear algebra:



$$\{ \text{ Orders } \} = \{ \text{ Half-planes } \} = \{ \text{ (Projectivized) Functionals } \}$$

We have actions of $GL_n(\mathbb{Z})$ on

$$Gr(n, n-1) = \{\text{co-dimension 1 hyperplanes}\} \text{ vs } \mathbf{P}_{>0}((\mathbb{R}^n)^*)$$

There is a Plücker embedding:

$$\mathbf{P}\left(\bigwedge^{n-1} \mathbb{R}^n\right) \text{ vs } \mathbf{P}_{>0}((\mathbb{R}^n)^*)$$

To deal away with projectivization, one uses work of Ronnie and Alekos on structurable equivalence relations.

We have actions of $GL_n(\mathbb{Z})$ on

$$\bigwedge^{n-1} \mathbb{R}^n \text{ vs } (\mathbb{R}^n)^*$$

The question becomes: What are the actions induced by $GL_n(\mathbb{Z}) \curvearrowright \mathbb{R}^n$? These two are isomorphic to $\mathbb{R}^n$ as vector spaces.

For dual: Act by the adjoint/transpose through the isomorphism taking $v \mapsto \langle v, \cdot \rangle$.

There is a basis for $\bigwedge^{n-1}\mathbb{R}^n$ given by $\widehat{e}_k = e_1 \wedge \ldots \wedge e_{k-1} \wedge e_{k+1} \wedge \ldots \wedge e_n$.

Calculation ...

If $M \in GL_n(\mathbb{Z})$,

$$\widetilde{M}\,\widehat{e}_i = \ldots + \det(M_{ij})\widehat{e}_j + \ldots$$

where M_{ij} is the matrix with ith row and jth column removed.

So the induced action is almost by the "classical adjoint"!

$$adj\ M = \left((-1)^{1+i+j}\det M_{ij}\right)$$

so once check that the classical adjoint is "surjective", one gets the same equivalence relation that $E(GL_n(\mathbb{R}) \curvearrowright \mathbb{R}^n)$, so one studies that.

Thank you!