Quasi-treeable equivalence relations

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Setting

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Theorem (classical)

A f.g. group Γ is virtually free iff it has a Cayley graph G which is a **quasi-tree**.

Quasi-tree $\leftarrow$ graph quasi-isometric to a tree $\exists f : G \to T$ which

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There are $M > 1, K > 0$ s.t.

$$\frac{1}{M} d_T(f(x), f(y)) - K \leq d_G(x, y) \leq M d_T(f(x), f(y)) + K,$$
$$d_T(\mathrm{im}(f), z) \leq K.$$

for all $x, y \in V(G)$ and $z \in V(T)$.

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No, for bad reasons.

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Better Question

If a CBER is **l.f.** quasi-treeable, must it be treeable?

Results

Theorem (R. Chen, A. P., R. Tao, A. Tserunyan 2023+)

Let $E \subseteq X^2$ be a CBER, $G \subseteq E$ be a locally finite graphing whose each component is (abstractly) a quasi-tree.

(i) *If G is one-ended, then E is hyperfinite.*

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Manning's bottleneck criterion (2005)

*A graph $G \subseteq X^2$ is a quasi-tree iff there exists $K > 0$ s.t. every $x, y \in X$ have an K-**approx. midpoint** m:*

$$d(x, m) \approx \tfrac{1}{2} d(x, y) \approx d(m, y),$$

$$x, y \text{ lie in different components of } B(m, K)^c.$$

Proof: The one-ended case

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Let $G \subseteq X^2$ be a locally finite one-ended quasi-tree. Then G is hyperfinite.

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Fix K witnessing Manning's bottleneck criterion. For $Q > 0$, call $x \in X$ a Q-**leaf** if

$$B(x, 2K) \cup \text{finite components of } X \setminus B(x, 2K) \subseteq B(x, Q).$$

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Lemma

- *Every $x \in X$ is a Q-leaf for some Q.*
- $X_Q \nearrow X$ *and* $E(G \restriction X_Q) \nearrow E(G)$.
- *Each connected component of X_Q has diameter $\leq 2Q + 4K$.*

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- WLOG x is in a finite component of $B(m', 2K)^c$.
- But $d(x, m') \geq \frac{1}{2}d(x, y) - 2K > \frac{1}{2}(2Q + 4K) - 2K = Q$.
- m' cannot possibly be a Q-leaf. #

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Fix K witnessing bottleneck criterion. Consider the set of oriented cuts

$$\mathcal{C} = \{A \subseteq X \mid A, \neg A \text{ connected } \& \text{ diam}(\partial A) \leq 2K + 2\} \cup \{\varnothing, X\}.$$

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Lemma

Each $\varnothing, X \neq A \in \mathcal{C}$ is adjacent to $\leq 3^{D^{12K+10}}$ other $\varnothing, X \neq B \in \mathcal{C}$.

$A, B \in \mathcal{C}$ are **adjacent** if they "cross" or don't cross but there's "no cut in between".

Proof: The bounded degree case

We reduce G to the following graph $\widehat{G}$:

$$V(\widehat{G}) = \{U \subset \mathcal{C} : U \text{ is a clopen ultrafilter}\}$$

Every such ultrafilter has a $\subseteq$-minimal elements, which must be adjacent, hence there is a uniform bound.

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Cycle cut using the finite coloring and geometry of half-planes.

Thank you!