Quasi-treeable equivalence relations

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Background - CBERs

Interested in studying classification problems and invariants.

Throughout, X is a standard Borel space, such as the interval $[0,1]$ or the Cantor space $2^{\mathbf{N}}$.

Countable Borel equivalence relations

A **countable Borel equivalence relation (CBER)** is an equivalence relation E which:

- is a Borel subset of X^2.
- has countable classes.

Background - Examples of CBERs

1. The identity relation on X $=_X$ is a countable Borel equivalence relation.
2. Eventual equality and tail equivalence on sequences $(x_i) \in S^{\mathbf{N}}$:
$$(x_i)\ E_0\ (y_j) \longleftrightarrow \exists N, \forall n \geq N, x_n = y_n$$
$$(x_i)\ E_t\ (y_j) \longleftrightarrow \exists N, m, \forall n \geq N, x_n = y_{n+m}$$
3. Orbit equivalence relations of Borel actions of countable groups $\Gamma \curvearrowright X$:
$$x\ E(\Gamma \curvearrowright X)\ y \longleftrightarrow \exists \gamma \in \Gamma, \gamma x = y.$$

Theorem [Feldman-Moore, '77]

All countable Borel equivalence relations arise as orbit equivalence relations.

Background - Reductions

Reductions

If $(X, E), (Y, F)$ are two CBERs, a Borel function $f : X \to Y$ such that

$$x\, E\, y \longleftrightarrow f(x)\, F\, f(y)$$

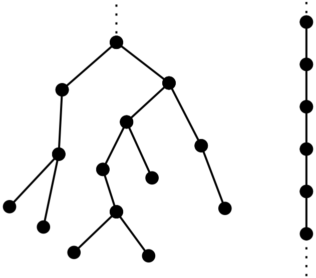
is called a **reduction**. We write $E \leq F$.

$$=_X \quad < \quad E_0, E_t, E(\mathbf{Z} \curvearrowright X) \quad < \quad E\left(F_2 \curvearrowright 2^{F^2}_{free}\right) \quad < \quad E\left(SL_3(\mathbf{Z}) \curvearrowright 2^{SL_3(\mathbf{Z})}_{free}\right)$$

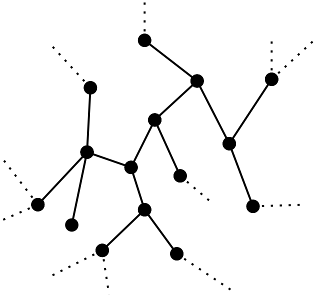
smooth $<$ **hyperfinite** $<$ **treeable** $<$ **(non-treeable)**

Structurability of CBERs

$\leq$



$\leq$



SMOOTH
Natural line (or finite)

HYPERFINITE
One or two ended trees

TREEABILITY
Arbitrary trees

Background - Quasi-trees

Quasi-tree $\leftarrow$ graph quasi-isometric to a tree $\exists f : G \to T$ which

- f roughly preserves distances,
- f is roughly surjective.

There are $M > 1, K > 0$ s.t.

$$\frac{1}{M} d_T(f(x), f(y)) - K \leq d_G(x, y) \leq M d_T(f(x), f(y)) + K,$$

$$d_T(\mathrm{im}(f), z) \leq K.$$

for all $x, y \in V(G)$ and $z \in V(T)$.

Motivation - Dynamics

Γ free, $\Gamma \curvearrowright X$ free $\implies E(\Gamma \curvearrowright X)$ treeable.

Γ virtually free, $\Gamma \curvearrowright X$ free $\implies E(\Gamma \curvearrowright X)$ quasi-treeable, i.e there exists some graphing whose connected components are quasi-trees.

Theorem (Follows from Jackson–Kechris–Louveau '02)

Γ *virtually free*, $\Gamma \curvearrowright X$ *free* $\implies E(\Gamma \curvearrowright X)$ *treeable.*

Better Question

If a CBER is **l.f.** quasi-treeable, must it be treeable?

No, for bad reasons. Requires l.f.

Results

Theorem (R. Chen, A. P., R. Tao, A. Tserunyan 2023+)

Let $E \subseteq X^2$ be a CBER, $G \subseteq E$ be a locally finite graphing whose each component is a quasi-tree.

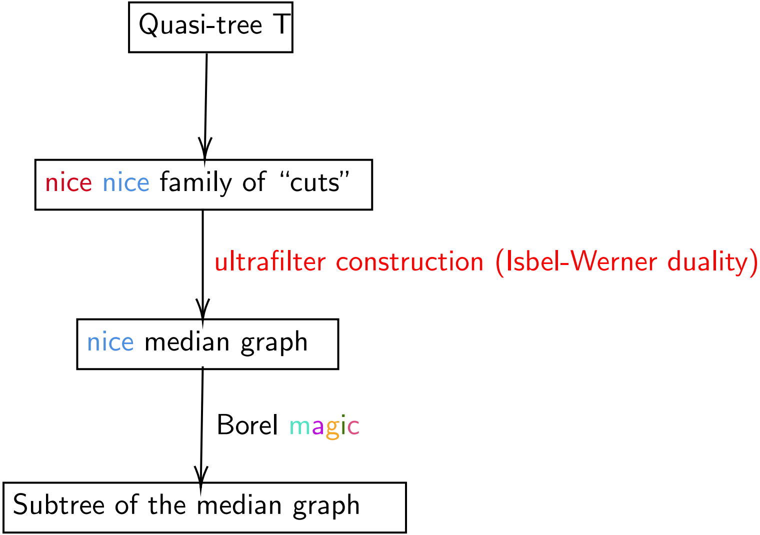
(i) *G is treeable.*

(ii) *If G is one-ended, then E is hyperfinite.*

Theorem (R. Chen, A. P., R. Tao, A. Tserunyan 2023+)

Let $E \subseteq X^2$ be a CBER, $G \subseteq E$ be a locally finite graphing whose each component is a quasi-tree. If G has a global bound on degree, there is a reduction to a Borel tree $(Y, \mathcal{T})$ which is a quasi-isometry (class-wise).

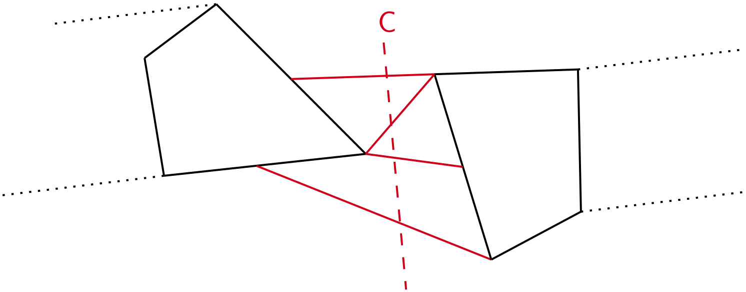
Overview of the proof



Cuts

Given a countable graph G, the set of (connected) cuts of G is

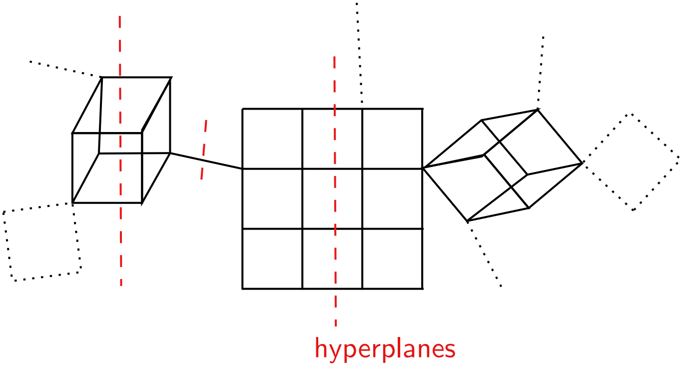
$$\mathcal{C}(G) := \{C \Subset E(G) : G - C \text{ has 2 connected components}\}$$



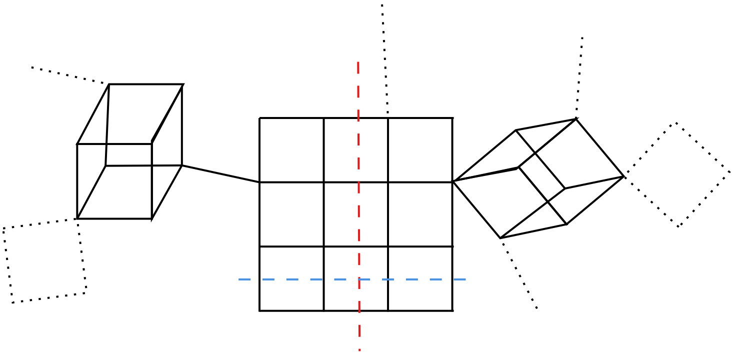
The two components of $G - C$ are called the **sides** of C.

Median graphs

A **median graph** can always be represented as 1-skeleton of CAT(0) cube complexes.



Perpendicular hyperplanes



perpendicular hyperplanes

Hyperplanes are **perpendicular** if all pair of sides intersect.

Isbel-Werner duality

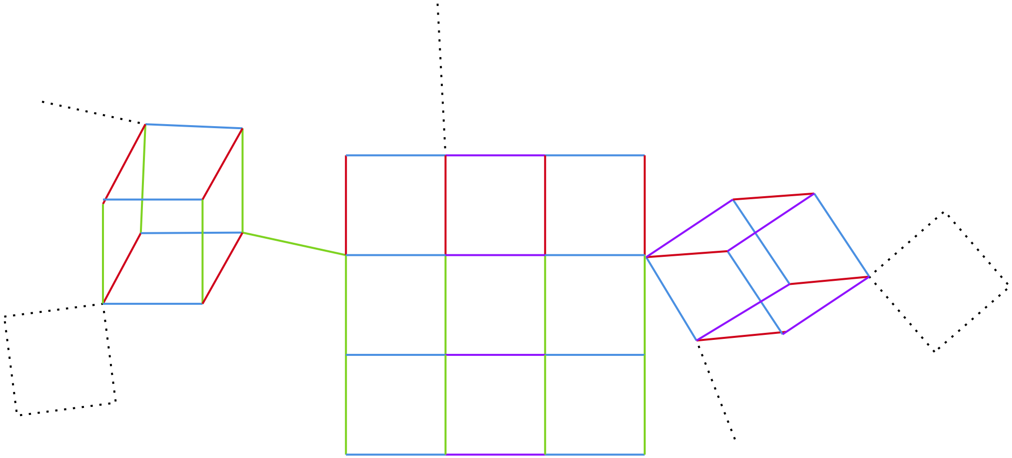
$\mathcal{C}_R(T)$	Median graph $\mathcal{O}_R(T)$
Ultrafilter of cuts	Vertices
Ultrafilters differing on a single cut	edge
Cuts	Hyperplanes
Crossing Cuts	Perpendicular hyperplanes
Finite number of cuts in a finite window	Hyperplanes contain finitely many edges
Ends are separated	Finite-to-1 map $T \to \mathcal{O}_R(T)$

Lemma (Follows from Kechris-Miller '04)

There exists a countable coloring of hyperplanes such that if two hyperplanes are perpendicular, they have different color.

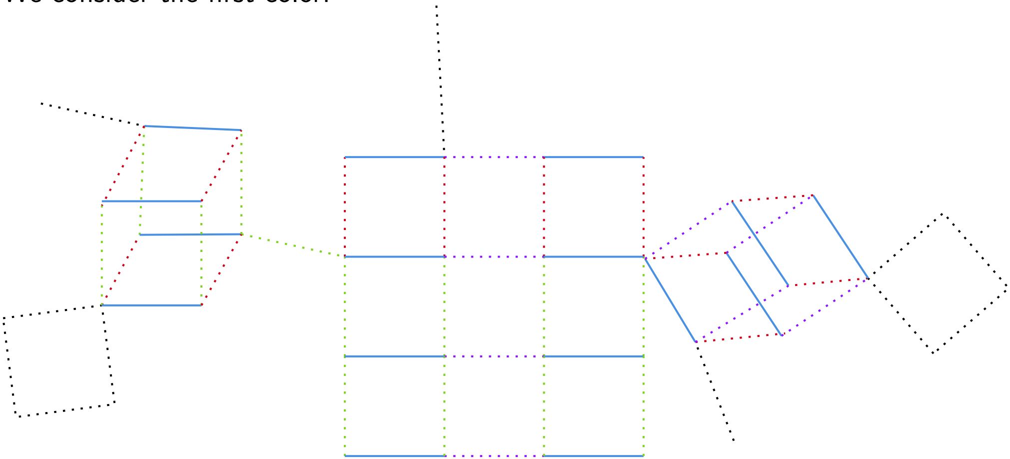
Colorings

We have a coloring now:



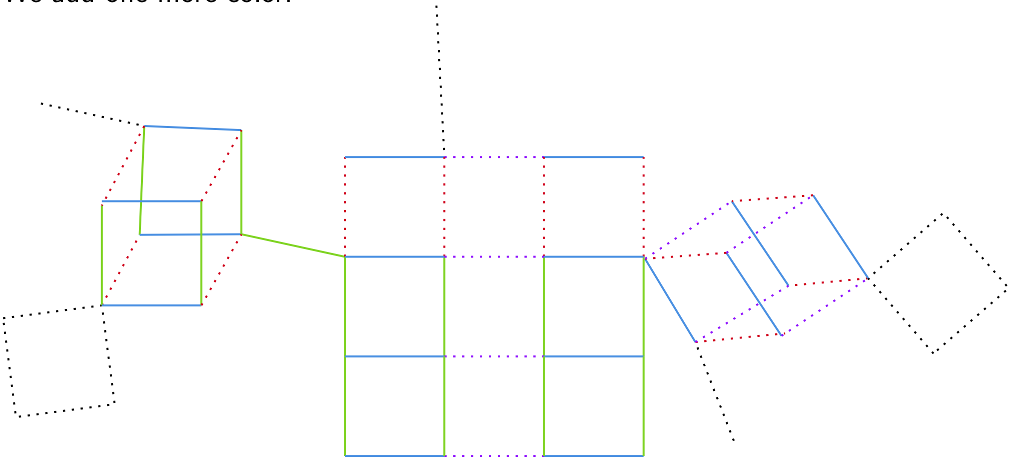
Building the tree: first color

We consider the first color:



Adding more colors

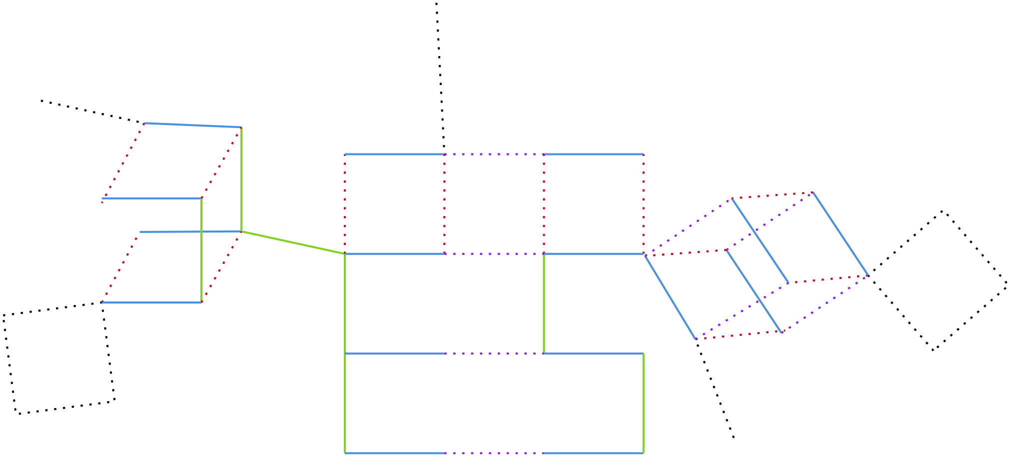
We add one more color:



But we don't have a tree anymore!

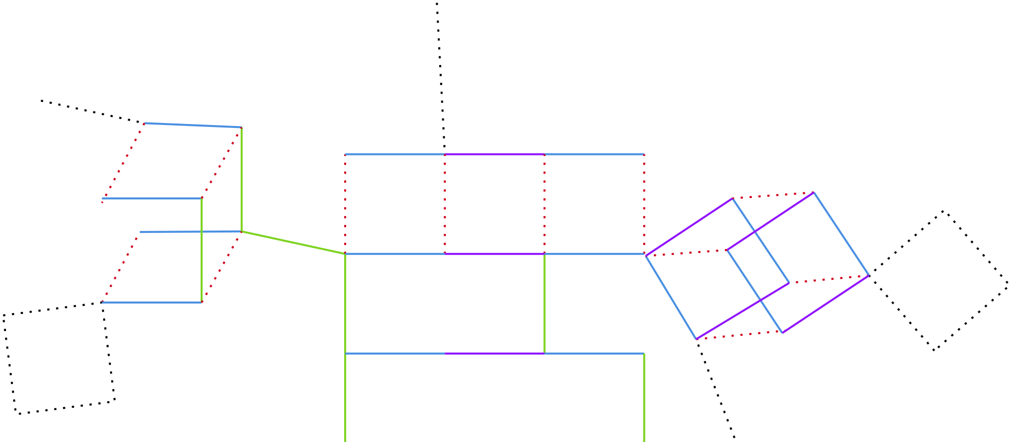
Cycle cutting

For every hyperplane, we keep only the minimal amount of edges which preserves connectedness.



Iterative procedure

Then we go again!



Iterative procedure

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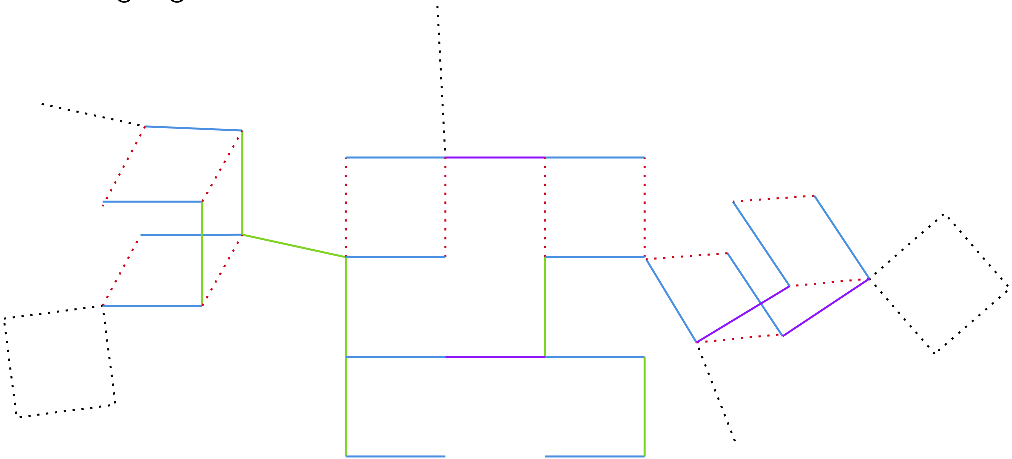
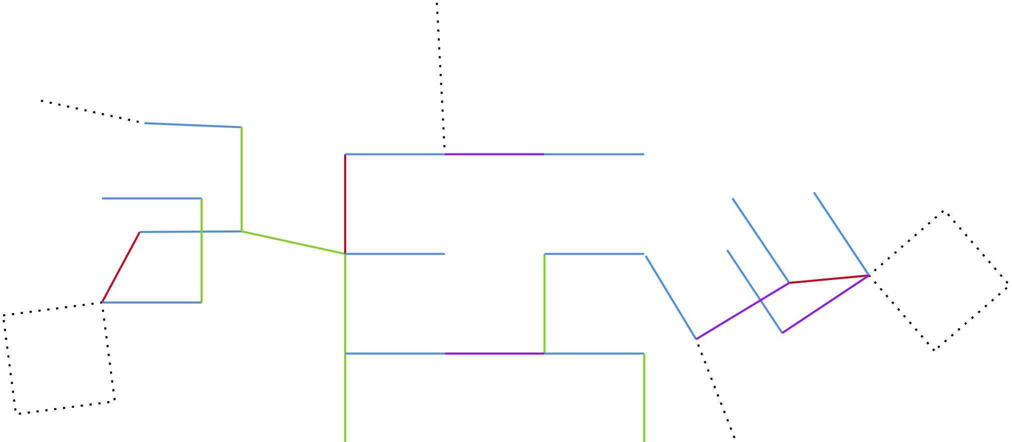


After 4 colors

Skipping 2 steps:



After all is said and done

Theorem (R. Chen, A. P., R. Tao, A. Tserunyan 2023+)

If $\mathcal{O}$ is a median graph with a countable coloring of hyperplanes such that perpendicular hyperplanes have different colors, there is a "canonical" subtree $\widehat{T} \subset \mathcal{O}$.

Can be generalized to other "tree-like" notions for graph:

Theorem (R. Chen, A. P., R. Tao, A. Tserunyan 2023+)

If a CBER E admits a locally finite graphing with components quasi-trees or of bounded tree-width, then E is treeable.

Thank you!