

EXACTNESS AND THE TOPOLOGY OF THE SPACE OF INVARIANT RANDOM EQUIVALENCE RELATIONS

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ABSTRACT. We characterize exactness of a countable group Γ in terms of invariant random equivalence relations (IREs) on Γ . Specifically, we show that Γ is exact if and only if every weak limit of finite IREs is an amenable IRE. In particular, for exact groups this implies amenability of the restricted rerooting relation associated to the ideal Bernoulli Voronoi tessellation, the discrete analog of the ideal Poisson Voronoi tessellation.

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1. INTRODUCTION

Invariant random equivalence relations (IREs) on groups were first introduced by Tucker-Drob [TD13] and later studied by Kechris [Kec17]. Let Γ be a countable discrete group and let $\mathcal{E} \subset 2^{\Gamma \times \Gamma}$ denote the standard Borel space of equivalence relations on Γ . An **invariant random equivalence relation (IRE)** is a Borel probability measure ρ on \mathcal{E} which is invariant under the left shift action $\Gamma \curvearrowright \mathcal{E}$. A **finite IRE** is an IRE ρ such that ρ -almost every equivalence relation has only finite classes.

In this paper, we primarily concern ourselves with amenability of IREs. An **amenable IRE** is an IRE ρ for which there exist Borel maps $\lambda_n: \mathcal{E} \rightarrow \text{Prob}(\Gamma)$ such that for ρ -almost every $r \in \mathcal{E}$ and for every $g \in [e]_r$,

$$\|g^{-1}.\lambda_n^r - \lambda_n^{g^{-1}.r}\|_1 \rightarrow 0.$$

For example, the *full* IRE $\delta_{\Gamma \times \Gamma}$ is amenable if and only if Γ is amenable.

An immediate consequence of Kechris [Kec17, Theorem 15.7] is that the full IRE on a group is amenable if and only if it is a weak limit of finite IREs. In fact, every amenable IRE is a weak limit of finite IREs (see Proposition 3.10). Conversely, one might also expect that weak limits of finite IREs are amenable in general. Perhaps surprisingly, this expectation fails for every nonexact group.

Theorem 1.1 (see Theorem 5.1). Let Γ be a countable discrete group. If Γ is nonexact, then there exists a weak limit of finite IREs on Γ which is not amenable.

Exactness of groups is a weakening of amenability, introduced independently by Kirchberg and Wassermann [KW99] and by Yu [Yu00] (for more on exactness, see [AD07]). A group Γ is **exact** if and only if it admits a continuous action on a compact space which is topologically amenable [AD02, GK02, HR00, Oza00]. For a finitely generated group, exactness is equivalent to uniform local amenability of its Cayley graphs [BNv⁺13, Ele21]. Therefore bounded degree Cayley graphs of nonexact groups admit an embedded sequence of small scale expanders (see Definition 2.2). We prove Theorem 1.1 by constructing finite IREs on Γ whose classes are close enough to small scale expanders so that the weak limit cannot be amenable.

On the other hand, under the assumption of exactness, the initial expectation about amenability of weak limits of finite IREs is realized.

Theorem 1.2 (see Theorem 4.5). Let Γ be a countable discrete group. If Γ is exact, then every weak limit of finite IREs on Γ is amenable.

A topologically amenable action on a compact space can be thought of as a boundary of Γ . The key idea in the proof of Theorem 1.2 is that any weak limit ρ of finite IREs on an exact group admits a marking of the classes of ρ -almost every $r \in \mathcal{E}$ by such an action of the group Γ . This marking can be thought of as a selection of a boundary point for each class of r . The existence of such a marking then implies that ρ is an amenable IRE.

Let IRE_Γ , AIRE_Γ , and FIRE_Γ denote the spaces of all, amenable, and finite IREs on Γ respectively, endowed with the weak convergence topology. Since for every group Γ every amenable IRE on Γ is a weak limit of finite IREs, Theorems 1.1 and 1.2 give a characterization of exactness in terms of IREs, à la Kechris's characterization [Kec17, Theorem 15.7] of amenability being equivalent to $\overline{\text{FIRE}}_\Gamma = \text{IRE}_\Gamma$.

Corollary 1.3. A countable discrete group is exact if and only if $\overline{\text{FIRE}}_\Gamma = \text{AIRE}_\Gamma$.

Additional motivation. The original impetus for this work was to better understand the proof of fixed price one for higher rank semisimple Lie groups [FMW23]. There, the main object of study is the ideal Poisson Voronoi tessellation (IPVT) of a locally compact second countable group G . The IPVT is a weak limit of IREs with finite volume classes – it arises as the low-intensity limit of the Voronoi tessellations associated to Poisson point processes on G . A key step in [FMW23] is proving an amenability property of this IRE. It is a natural question to ask whether amenability is a general consequence of being a limit of IREs with finite volume classes. In Theorems 1.1 and 1.2 we fully resolve the discrete version of this question.

The first studied examples of the IPVT may be found in the PhD thesis of Bhupatiraju [Bhu10], where the model is studied on regular trees and the hyperbolic plane. The concept also appears in [BCP22]. Existence of the IPVT in hyperbolic spaces was shown in [DCE⁺23], and independently for symmetric spaces in [FMW23]. Recently, it was also shown in [D'A24] to exist on $\mathbb{H}^2 \times \mathbb{H}^2$ with the ℓ_1 metric.

In Section 6 we pose some open questions on the discrete analogue of the IPVT: the ideal Bernoulli Voronoi tessellation.

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2. PRELIMINARIES

Throughout Γ will denote a countable discrete group. If $\Gamma \curvearrowright X$ is an action on a set X , we denote the action of $\gamma \in \Gamma$ on an element $x \in X$ by $\gamma.x$ unless otherwise stated. A **probability measure preserving (pmp)** action $\Gamma \curvearrowright (X, \mu)$ is an action on a standard probability space (X, μ) satisfying $\mu(A) = \mu(\gamma.A)$ for every measurable $A \subseteq X$ and $\gamma \in \Gamma$.

2.1. Exact groups. The reader is referred to [BO08, Section 4.2 and Chapter 5] for an introduction to topologically amenable actions and exact groups. We review the definitions necessary for our purposes here.

Let $\text{Prob}(\Gamma)$ denote the set of probability measures on Γ equipped with the topology inherited as a subspace of $\ell^1(\Gamma)$. Let $\Gamma \curvearrowright \text{Prob}(\Gamma)$ be the action given by the shift $\gamma.\eta(g) = \eta(\gamma^{-1}g)$, where $\gamma, g \in \Gamma$ and $\eta \in \text{Prob}(\Gamma)$.

Definition 2.1. An action of a countable group Γ on a topological space X by homeomorphisms is **topologically amenable** if there exists a sequence $(\eta_n)_{n \in \mathbb{N}}$ of continuous maps

$\eta_n: X \rightarrow \text{Prob}(\Gamma)$ such that for each $\gamma \in \Gamma$,

$$\sup_{x \in X} \|\gamma \cdot \eta_n^x - \eta_n^{\gamma \cdot x}\|_1 \rightarrow 0.$$

A countable group is **exact** if it admits a topologically amenable action on a compact metrizable space.

This is not one of the standard definitions of exactness as we include the additional assumption of metrizability. However, it is equivalent for countable groups by [BO08, Theorem 5.1.7]. This characterization of exactness is often referred to as *boundary amenability* or *amenability at infinity*.

Examples of nonexact groups were first produced by Gromov [Gro00, Gro03]. An alternative construction is due to Osajda [Osa20], who produced examples with an embedded sequence of expanders in their Cayley graphs. In Section 4 we will use the above definition of exactness in terms of topologically amenable actions, whilst in Section 5 we will make use of the combinatorial characterization of nonexactness given below.

Definition 2.2. A finite graph G is a **small scale** (κ, N) -**expander**, for some $\kappa > 0$ and $N \in \mathbb{N}$, if for every $F \subset V(G)$ with $|F| \leq N$ we have

$$\frac{|\partial_G F|}{|F|} \geq \kappa.$$

A sequence of finite graphs $(G_n)_{n \in \mathbb{N}}$ is a **sequence of small scale κ -expanders**, for some $\kappa > 0$, if each G_n is a small scale (κ, n) -expander. We say that $(G_n)_{n \in \mathbb{N}}$ is a **sequence of small scale expanders** if there exists some $\kappa > 0$ for which it is a sequence of small scale κ -expanders.

Theorem 2.3 (Brodzki–Niblo–Špakula–Willett–Wright [BNv+13], Elek [Ele21]). Let Γ be a finitely generated group with finite generating set S . Then Γ is nonexact if and only if the Cayley graph $\text{Cay}(\Gamma, S)$ contains a sequence of small scale expanders as induced subgraphs.

2.2. Borel and measured combinatorics. Let X be a standard Borel space. Recall that a **countable Borel equivalence relation (cber)** on X is a Borel subset $\mathcal{R} \subseteq X \times X$ defining an equivalence relation on X with countable equivalence classes. If the relation has finite classes, we say it is a finite Borel equivalence relation.

A **probability measure preserving (pmp) cber** on a standard probability space (X, μ) is a cber \mathcal{R} such that $\int c_x d\mu = \int c^x d\mu$ where c_x and c^x are the counting measures on $\{x\} \times [x]_{\mathcal{R}}$ and $[x]_{\mathcal{R}} \times \{x\}$, respectively.

Definition 2.4. A cber \mathcal{R} on X is **Borel amenable** if there exists a sequence of **Hulanicki–Reiter functions**: a sequence of Borel maps $\eta_n: \mathcal{R} \rightarrow \mathbb{R}_{\geq 0}$ such that

- $\eta_n^x \in \text{Prob}([x]_{\mathcal{R}})$ for all $x \in X$ and $n \in \mathbb{N}$, where $\eta_n^x(y) := \eta_n(y, x)$ for every $y \in [x]_{\mathcal{R}}$, and

- $\|\eta_n^x - \eta_n^y\|_1 \rightarrow 0$ for every $(y, x) \in \mathcal{R}$.

A pmp cber \mathcal{R} on a standard probability space (X, μ) is μ -**amenable** if its restriction to a μ -conull \mathcal{R} -invariant Borel subset of X is Borel amenable. For more on amenability of equivalence relations, the reader is referred to [KM04] and [Moo20].

If $\Gamma \curvearrowright X$ is a Borel action of a countable group Γ on a standard Borel space X , we let $\mathcal{R}_{\Gamma \curvearrowright X}$ denote the **orbit equivalence relation**. This cber is defined by letting $(y, x) \in \mathcal{R}_{\Gamma \curvearrowright X}$ if and only if there exists $\gamma \in \Gamma$ such that $\gamma.x = y$, for any $x, y \in X$.

Lemma 2.5. Let $\Gamma \curvearrowright X$ be a Borel action on a standard Borel space and $\mathcal{S} \leq \mathcal{R}_{\Gamma \curvearrowright X}$ a Borel subequivalence relation. Suppose there exists a sequence of Borel maps $\eta_n: X \rightarrow \text{Prob}(\Gamma)$, for $n \in \mathbb{N}$, such that for every $x \in X$ and $\gamma \in \Gamma$ satisfying $(\gamma.x, x) \in \mathcal{S}$,

$$\|\gamma.\eta_n^x - \eta_n^{\gamma.x}\|_1 \rightarrow 0.$$

Then \mathcal{S} is Borel amenable.

Proof. Fix a Borel \mathcal{S} -invariant family of retracts $[x]_{\mathcal{R}_{\Gamma \curvearrowright X}} \rightarrow [x]_{\mathcal{S}}$ (for instance, by using Lusin–Novikov). Pushing forward η_n^x first to $\text{Prob}([x]_{\mathcal{R}_{\Gamma \curvearrowright X}})$, then to $\text{Prob}([x]_{\mathcal{S}})$ witnesses Borel amenability of \mathcal{S} . \square

A **Borel graph** on a standard Borel space X is a symmetric Borel subset $\mathcal{G} \subseteq X \times X$. Say the graph has **bounded degree** if there exists $d \in \mathbb{N}$ such that $\deg_{\mathcal{G}}(x) \leq d$ for every $x \in X$. For a vertex $x \in X$, let \mathcal{G}_x denote its connected component in \mathcal{G} . A bounded degree Borel graph \mathcal{G} induces a cber $\mathcal{R}_{\mathcal{G}}$ on X defined by taking the vertices of each connected component to be their own equivalence class, i.e., $x, y \in X$ are $\mathcal{R}_{\mathcal{G}}$ -equivalent if $\mathcal{G}_x = \mathcal{G}_y$. For more about combinatorics on Borel graphs, see [KM16].

Let (X, μ) be a standard probability space. A Borel graph \mathcal{G} on X is **probability measure preserving (pmp)** if $\mathcal{R}_{\mathcal{G}}$ is a pmp cber on (X, μ) . If \mathcal{R} is a pmp cber and \mathcal{G} a Borel graph on (X, μ) , we say that \mathcal{G} is a **graphing** of \mathcal{R} if $\mathcal{R}_{\mathcal{G}} = \mathcal{R}$ almost surely.

2.3. Graph convergence. In this subsection we recall some basic facts about Benjamini–Schramm convergence. The reader is referred to the survey paper of Aldous and Lyons [AL07] and the book of Lovasz [Lov12] for a thorough introduction to the topic.

Let \mathbb{G}_{\bullet} (resp. $\mathbb{G}_{\bullet\bullet}$) denote the set of isomorphism classes of rooted (resp. doubly rooted) countable connected graphs with degree bounded by $d \in \mathbb{N}$. For a rooted graph (G, u) , let $B_r(G, u)$ denote the ball of radius r in G about u with root at u . We endow \mathbb{G}_{\bullet} with the metric defined by

$$d((G, u), (H, v)) = \inf\{2^{-r} : B_r(G, u) \cong B_r(H, v)\},$$

for $(G, u), (H, v) \in \mathbb{G}_{\bullet}$. This metric makes \mathbb{G}_{\bullet} a compact zero-dimensional Polish space. A similarly defined metric gives rise to a Polish topology on $\mathbb{G}_{\bullet\bullet}$ with the same properties.

A **unimodular random graph** is a random rooted graph (G, o) of \mathbb{G}_\bullet that satisfies the Mass Transport Principle: for every Borel map $f: \mathbb{G}_{\bullet\bullet} \rightarrow \mathbb{R}_{\geq 0}$,

$$\mathbb{E} \left[\sum_{v \in V(G)} f(G, v, o) \right] = \mathbb{E} \left[\sum_{v \in V(G)} f(G, o, v) \right].$$

Example 2.6 (Cayley graphs). An example of a unimodular random graph is the Cayley graph of a finitely generated group rooted at the identity $e \in \Gamma$. In this specific case, the mass transport principle takes the following form. If $F: \Gamma \times \Gamma \rightarrow \mathbb{R}_{\geq 0}$ is diagonally invariant, then

$$\sum_{\gamma \in \Gamma} F(e, \gamma) = \sum_{\gamma \in \Gamma} F(\gamma, e).$$

We will apply this to functions F of the form $F(x, y) = \mathbb{E}[f(x, y; \Pi)]$, where Π is a Γ -invariant stochastic process and f is diagonally invariant (that is, $f(\gamma x, \gamma y; \gamma \cdot \Pi) = f(x, y; \Pi)$).

Example 2.7. Let \mathcal{G} on (X, μ) be a pmp bounded degree Borel graph. Then the random variable $x \in X \mapsto (\mathcal{G}_x, x) \in \mathbb{G}_\bullet$ is a unimodular random graph.

Example 2.8. A particular instance of the latter construction is the unimodular random graph associated to a finite graph G , defined as $(G, v) \in \mathbb{G}_\bullet$, where $v \in V(G)$ is sampled from the uniform distribution on $V(G)$.

A sequence of unimodular random graphs (G_n, o) **Benjamini–Schramm converges** to a unimodular random graph (G, o) if $B_r(G_n, o)$ converges in distribution to $B_r(G, o)$ for every $r \in \mathbb{N}$.

2.4. Hyperfiniteness. In this subsection we discuss hyperfiniteness of pmp cbers and unimodular random graphs.

A cber \mathcal{R} on X is **hyperfinit** if there exists an increasing union of finite Borel subequivalence relations $(\mathcal{F}_n)_n$ of \mathcal{R} such that $\mathcal{R} = \bigcup_n \mathcal{F}_n$. A pmp cber \mathcal{R} on (X, μ) is **μ -hyperfinit** if the restriction of \mathcal{R} to a μ -conull \mathcal{R} -invariant Borel subset is hyperfinite. The classic article of Connes–Feldman–Weiss [CFW81] proves that μ -amenability and μ -hyperfiniteness are equivalent. The analogous purely Borel statement remains an important open question. In this paper, we will use the μ -amenable perspective in Section 4 and the μ -hyperfinit perspective in Section 5.

A unimodular random graph (G, o) is **(ε, k) -hyperfinit** for $\varepsilon > 0$ and $k \in \mathbb{N}$ if there exists a coupling (G, E, o) with $E \subset E(G)$ such that

- (1) $\mathbb{E}[\deg_E(o)] \leq \varepsilon$ and
- (2) $G - E$ has connected components of size at most k almost surely.

Say that a unimodular random graph (G, o) is **hyperfinit** if for every $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that (G, o) is (ε, k) -hyperfinit.

We will make use of the following well-known connection between these notions which follows immediately from [Ele12, Proposition 2.2] and [CGMTD21, Theorem 1.1].

Proposition 2.9. Let \mathcal{R} be a pmp cber on (X, μ) and \mathcal{G} a bounded degree graphing of \mathcal{R} . Then \mathcal{R} is μ -hyperfinite if and only if the unimodular random graph (\mathcal{G}_x, x) is hyperfinite.

Schramm proved the following theorem witnessing hyperfiniteness of a limit in the converging sequence.

Theorem 2.10 (Schramm [Sch08, Theorem 1.2]). Let (G_n, o) be a sequence of unimodular random graphs Benjamini–Schramm converging to (G, o) . If (G, o) is hyperfinite, then for every $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that (G_n, o) is (ε, k) -hyperfinite for n large enough.

Finally, we define the notion of a finite graph being (ε, k) -hyperfinite. We say that a finite graph G is (ε, k) -**hyperfinite** if there exists a subset of edges $E \subset E(G)$ with $|E| \leq \varepsilon|V(G)|$ such that $G \setminus E$ has connected components of size at most k . Hyperfiniteness of G in this sense is equivalent to hyperfiniteness of the associated unimodular random graph (G, v) [Sch08, Lemma 1.3].

3. INVARIANT RANDOM EQUIVALENCE RELATIONS

Let Γ be a countable group. Denote by \mathcal{E}_Γ the set of equivalence relations on Γ and equip it with the topology inherited as a subspace of $\{0, 1\}^{\Gamma \times \Gamma}$. The group Γ acts on \mathcal{E}_Γ by the left shift, i.e., $\gamma.r(x, y) = r(\gamma^{-1}x, \gamma^{-1}y)$ for $\gamma, x, y \in \Gamma$. We will simply often simply write \mathcal{E} , instead of \mathcal{E}_Γ .

Definition 3.1. An **invariant random equivalence relation (IRE)** is a Borel probability measure ρ on \mathcal{E} which is invariant under the action $\Gamma \curvearrowright \mathcal{E}$.

The space $\text{Prob}(\mathcal{E})$ of Borel probability measures on \mathcal{E} is a compact metrizable space when endowed with the weak topology. Denote by $\text{IRE}_\Gamma \subset \text{Prob}(\mathcal{E})$ the subset of IREs. Note that IRE_Γ is a closed subset of the sequentially compact space $\text{Prob}(\mathcal{E})$, so it is sequentially compact too. IREs have previously been studied by Tucker-Drob [TD13] and Kechris [Kec17].

Example 3.2 (Cosets). If $\Lambda \leq \Gamma$ is a subgroup, then its left coset decomposition Γ/Λ defines an IRE supported on a single element of \mathcal{E} . The two extremes of $\Lambda = \{1\}$ and $\Lambda = \Gamma$ give the equality IRE δ_Δ , where Δ denotes the diagonal, and the full IRE $\delta_{\Gamma \times \Gamma}$ respectively. Given an invariant random subgroup $\Lambda \leq \Gamma$, the right coset decomposition $\Lambda \setminus \Gamma$ is also an IRE.

Example 3.3 (Percolation). Given an invariant percolation process on Γ (such as Bernoulli bond percolation), we may turn it into an IRE by associating to each configuration the relation of being in the same connected component.

Example 3.4. Given a pmp action $\Gamma \curvearrowright (X, \mu)$, a subequivalence relation $\mathcal{S} \leq \mathcal{R}_{\Gamma \curvearrowright X}$ naturally produces an IRE by pulling back $\mathcal{S}|_{[x]_{\mathcal{R}} \times [x]_{\mathcal{R}}}$ to Γ , and every IRE arises in this fashion (see Proposition 8.3 of [TD13] and Proposition 15.3 of [Kec17]).

In the following proposition we relate Benjamini–Schramm convergence of graphs to weak convergence of IREs, so we opt for a random variable approach. In a slight abuse of terminology, we say that a \mathcal{E} -valued random variable R is an IRE if its distribution is an IRE. Let \mathcal{C} be a Cayley graph of Γ . If R is an IRE, we let $\mathcal{C}_{\text{con}}(o, R)$ denote the connected component of the identity e in the subgraph of \mathcal{C} induced by $[e]_R$ and rooted at $o = e$. It is easy to see that $\mathcal{C}_{\text{con}}(o, R)$ is unimodular when R is an IRE.

Proposition 3.5. Let Γ be a finitely generated group with bounded degree Cayley graph \mathcal{C} and R_n a sequence of IREs on Γ converging in distribution to an IRE R . Then the unimodular random graphs $\mathcal{C}_{\text{con}}(o, R_n)$ Benjamini–Schramm converge to $\mathcal{C}_{\text{con}}(o, R)$.

Proof. Denote $(G_n, o) := \mathcal{C}_{\text{con}}(o, R_n)$ and $(G, o) := \mathcal{C}_{\text{con}}(o, R)$ in order to ease notation. Since all the unimodular random graphs considered are supported on induced subgraphs of \mathcal{C} , it suffices to show that

$$\mathbb{P}[B_r(G_n, o) \cong (H, e)] \longrightarrow \mathbb{P}[B_r(G, o) \cong (H, e)]$$

for every $r \geq 0$ and every finite connected induced subgraph $H \leq \mathcal{C}$ containing the identity.

Fix such an r and subgraph H and let \mathcal{F} be the family of subsets $F \subseteq B_r(\mathcal{C}, e)$ such that $(\mathcal{C}[F], e) \cong (H, e)$. Then

$$\mathbb{P}[B_r(G_n, o) \cong (H, e)] = \mathbb{P}\left[\bigcup_{F \in \mathcal{F}} \{\text{conn. comp. of } o \text{ in } \mathcal{C}[[e]_{R_n} \cap B_r(\mathcal{C}, e)] \text{ is } F\}\right],$$

and similarly for (G, o) . The evaluated event is a finite union of clopen sets, so is clopen in \mathcal{E} . The conclusion follows. \square

3.1. Finite and amenable IREs and restricted rerooting. Let us now formally introduce the main objects of interest in this paper.

Definition 3.6. A **finite invariant random equivalence relation (FIRE)** is an IRE ρ such that ρ -almost every $r \in \mathcal{E}$ has all classes finite.

An **amenable invariant random equivalence relation (AIRE)** is an IRE ρ for which there exist Borel maps $\lambda_n: \mathcal{E} \rightarrow \text{Prob}(\Gamma)$ such that for ρ -almost every $r \in \mathcal{E}$ and for every $g \in [e]_r$,

$$\|g^{-1} \cdot \lambda_n^r - \lambda_n^{g^{-1} \cdot r}\|_1 \rightarrow 0.$$

Denote by FIRE_Γ (resp. AIRE_Γ) the subset of IRE_Γ consisting of finite (resp. amenable) IREs.

Remark 3.7. It is easy to see that every IRE on an amenable group is amenable by pushing forward the Reiter sequence on the group to \mathcal{E} . The converse statement that nonamenable groups admit nonamenable IREs is also true, but less immediate.

These definitions originally arose from studying the restricted rerooting relation.

Definition 3.8. The **restricted rerooting relation** is a Borel subequivalence relation $\mathcal{R}_{\text{re}} \leq \mathcal{R}_{\Gamma \curvearrowright \mathcal{E}}$. It is defined by

$$\mathcal{R}_{\text{re}} := \{(r, \gamma^{-1}.r) \in \mathcal{E} \times \mathcal{E} : \gamma \in [e]_r\}.$$

Call an IRE ρ is *free* if the action $\Gamma \curvearrowright (\mathcal{E}, \rho)$ is ρ -essentially free. A free IRE ρ is finite (resp. amenable) if and only if \mathcal{R}_{re} is finite ρ -almost everywhere (resp. ρ -amenable).

Remark 3.9. In general, for nonfree IREs the rerooting relation may be amenable while the IRE is not amenable, e.g., the full IRE $\delta_{\Gamma \times \Gamma}$ on a nonamenable group. One can maintain a meaningful connection to amenability by instead considering the *restricted rerooting groupoid* $\mathcal{G}_{\text{re}} := \{(\gamma, r) \in \Gamma \times \mathcal{E} : \gamma \in [e]_r\}$ which inherits its groupoid structure as a subgroupoid of the action groupoid $\Gamma \times \mathcal{E}$. Now, an IRE ρ is amenable if and only if \mathcal{G}_{re} is ρ -amenable as a discrete pmp groupoid. We won't make use of this characterization directly, but it motivates several propositions in this paper.

In this paper, we are primarily interested in the relationship between $\overline{\text{FIRE}}_\Gamma$ and AIRE_Γ . We now prove the general fact that $\text{AIRE}_\Gamma \subseteq \overline{\text{FIRE}}_\Gamma$ for every countable group Γ .

Proposition 3.10. Let Γ be a countable group. Every AIRE on Γ is a weak limit of FIREs.

Proof. Let ρ be an amenable IRE on Γ and let $\Gamma \curvearrowright (Z, \nu)$ be an essentially free pmp action. Consider the diagonal action $\Gamma \curvearrowright (Z \times \mathcal{E}, \nu \times \rho)$. Define a cber $\mathcal{R} \leq \mathcal{R}_{\Gamma \curvearrowright Z \times \mathcal{E}}$ by letting $((x, r), (y, t)) \in \mathcal{R}$ if there exists $\gamma \in [e]_r$ such that $(\gamma^{-1}.x, \gamma^{-1}.r) = (y, t)$.

Lemma 2.5 implies that \mathcal{R} is $\nu \times \rho$ -amenable since ρ is an AIRE. By [CFW81], we may find an increasing sequence $(\mathcal{F}_n)_{n \in \mathbb{N}}$ of finite Borel subequivalence relations of \mathcal{R} such that $\mathcal{R} = \bigcup_n \mathcal{F}_n$ up to a null set. Define $\sigma_n: Z \times \mathcal{E} \rightarrow \mathcal{E}$ by letting $(\gamma, \lambda) \in \sigma_n(z, r)$ if $(\gamma^{-1}.(z, r), \lambda^{-1}.(z, r)) \in \mathcal{F}_n$ for $\gamma, \lambda \in \Gamma$ and $(z, r) \in Z \times \mathcal{E}$. It is easy to see that the σ_n are Borel, Γ -equivariant, and that $(\sigma_n)_*(\nu \times \rho)$ is a FIRE.

For almost every $(z, r) \in Z \times \mathcal{E}$ we have that $\sigma_n(z, r) \rightarrow r$ in the weak topology. As random variables, it follows that $\sigma_n \rightarrow P$ converges almost surely and hence in distribution where $P: Z \times \mathcal{E} \rightarrow \mathcal{E}$ denotes the projection to the \mathcal{E} -coordinate. This concludes the proof as $P_*(\nu \times \rho) = \rho$. \square

4. IRES ON EXACT GROUPS

Proposition 3.10 showed that AIREs arise as weak limits of FIREs in general. The goal of this section is to prove the reverse containment $\overline{\text{FIRE}}_\Gamma \subseteq \text{AIRE}_\Gamma$ for exact groups (Theorem 4.5). We now give a sketch of the proof.

Suppose the group Γ is exact, and let $(\rho_n)_n$ be a sequence of FIREs weakly converging to an IRE ρ . Fix a topologically amenable action $\Gamma \curvearrowright X$ on a compact metric space. We use finiteness to produce a coupling of each ρ_n with an invariant random X -marking of Γ which has the property that the mark of any element of Γ determines equivariantly the mark of

any other restricted rerooting related element. This property is called “classwise coherence”. Any subsequential weak limit of these markings is still a classwise coherent invariant random X -marking for ρ . Finally, we use this marking to pull back amenability of the action $\Gamma \curvearrowright X$ to ρ itself.

In the definition below, we will consider the space Ξ^Γ where $\Gamma \curvearrowright \Xi$ is an action. We act on Ξ^Γ with the usual left shift action, that is,

$$(\gamma.\omega)(x) = \omega(\gamma^{-1}x) \text{ for all } \omega \in \Xi^\Gamma \text{ and } \gamma, x \in \Gamma.$$

Additionally, we let Γ act on $\mathcal{E} \times \Xi^\Gamma$ diagonally.

Definition 4.1. Let ρ be an IRE on Γ and Ξ a standard Borel space with a Borel action $\Gamma \curvearrowright \Xi$. A **classwise coherent** Ξ -marking of ρ is a Γ -invariant Borel probability measure $\tilde{\rho}$ on $\mathcal{E} \times \Xi^\Gamma$ such that

- the projection of $\tilde{\rho}$ to \mathcal{E} is ρ , and
- for $\tilde{\rho}$ almost every (r, f) and $(g, h) \in r$, we have $g.[f(g)] = h.[f(h)]$.

Note that if $\tilde{\rho}$ is a classwise coherent marking of ρ , then the mark of any $g \in \Gamma$ determines the mark at any other related element $h \in \Gamma$ as $f(h) = h^{-1}g.[f(g)]$. We call this property *local equivariance*.

Example 4.2. Let ρ be a FIRE and X a Γ -space. Fix a point x_0 in X . We construct a classwise coherent X -marking of ρ by marking a uniformly random point in each class by x_0 , and then marking all other points by the requirement of local equivariance. Explicitly, we can construct this measure as follows. Consider the (partially defined) map $\Phi : \mathcal{E} \times [0, 1]^\Gamma \rightarrow \mathcal{E} \times \Xi^\Gamma$ given by $\Phi(r, \omega) = (r, \varphi_{r, \omega})$, where

$$\varphi_{r, \omega}(g) = g^{-1}v_{g, r, \omega}.x_0,$$

and $v_{g, r, \omega}$ is the element of $[g]_r$ whose label (with respect to ω) is highest. Note that Φ is defined $\rho \otimes \text{Leb}^\Gamma$ almost surely. Then the desired classwise coherent X -marking is the pushforward measure $\Phi_*(\rho \otimes \text{Leb}^\Gamma)$.

The following proposition shows that an IRE that admits a classwise coherent marking by a topologically amenable action is itself amenable.

Proposition 4.3. Let ρ be an IRE on Γ , $\Gamma \curvearrowright \Xi$ a topologically amenable action, and $\tilde{\rho}$ a classwise coherent Ξ -marking of ρ . Then:

- (1) There exist Borel maps $\xi_n : \mathcal{E} \times \Xi^\Gamma \rightarrow \text{Prob}(\Gamma)$ such that for $\tilde{\rho}$ -almost every (r, f) , for every $g \in [e]_r$,

$$\|g^{-1}.\xi_n^{(r, f)} - \xi_n^{g^{-1}.\cdot, (r, f)}\|_1 \longrightarrow 0, \text{ and}$$

- (2) ρ is an AIRE.

Proof. For part (1), by definition of topological amenability there exists a sequence of continuous maps $\eta_n: \Xi \rightarrow \text{Prob}(\Gamma)$ such that for each $g \in \Gamma$,

$$\sup_{x \in \Xi} \|g \cdot \eta_n^x - \eta_n^{g \cdot x}\|_1 \longrightarrow 0.$$

We define $\xi_n: \mathcal{E} \times \Xi^\Gamma \rightarrow \text{Prob}(\mathcal{E})$ by $\xi_n^{(r,f)} := \eta_n^{f(e)}$. For every $g \in [e]_r$,

$$\|g^{-1} \cdot \xi_n^{(r,f)} - \xi_n^{g^{-1} \cdot (r,f)}\|_1 = \|g^{-1} \cdot \eta_n^{f(e)} - \eta_n^{f(g)}\|_1 = \|g^{-1} \cdot \eta_n^{f(e)} - \eta_n^{g^{-1} \cdot [f(e)]}\|_1 \longrightarrow 0$$

where we use local equivariance for $e \cdot [f(e)] = g \cdot [f(g)]$.

Part (2) follows from a standard argument as in, for instance [JKL02, Proposition 2.5 (ii)] or [Kec17, Proposition 17.9], which we include here.

Take the measure disintegration $(\rho_r)_{r \in \mathcal{E}}$ of $\tilde{\rho}$ over the projection $P: \mathcal{E} \times \Xi^\Gamma \rightarrow \mathcal{E}$, so that $\tilde{\rho} = \int_{\mathcal{E}} \rho_r d\rho(r)$ where $\rho_r \in \text{Prob}(\mathcal{E} \times \Xi^\Gamma)$ concentrates on $P^{-1}(r)$. By uniqueness of disintegration and invariance of $\tilde{\rho}$ and ρ , we have that $g_* \rho_r = \rho_{g \cdot r}$ for every $g \in \Gamma$ and ρ -almost every $r \in \mathcal{E}$.

For each $r \in \mathcal{E}$ and $n \in \mathbb{N}$, define $\alpha_n^r := \int_{\mathcal{E} \times \Xi^\Gamma} \xi_n^{(t,f)} d\rho_r(t, f) \in \text{Prob}(\Gamma)$. Observe that for almost every $r \in \mathcal{E}$, for every $g \in [e]_r$ we have

$$g^{-1} \cdot \alpha_n^r - \alpha_n^{g^{-1} \cdot r} = \int_{\mathcal{E} \times \Xi^\Gamma} g^{-1} \cdot \xi_n^{(t,f)} - \xi_n^{g^{-1} \cdot (t,f)} d\rho_r(t, f).$$

It follows from the dominated convergence theorem that for ρ -almost every $r \in \mathcal{E}$ such that $g \in [e]_r$, we have $\|g^{-1} \cdot \alpha_n^r - \alpha_n^{g^{-1} \cdot r}\|_1 \longrightarrow 0$. \square

Remark 4.4. In light of Remark 3.9, one may think of part (1) of Proposition 4.3 as proving Borel amenability of a certain extension of the restricted rerooting groupoid. The proof of part (2) of the proposition then shows that if a pmp extension of a groupoid is amenable, then the groupoid itself must be amenable.

We now have all the tools necessary for the proof of the main theorem of this section.

Theorem 4.5. Let Γ be an exact countable group. Then $\overline{\text{FIRE}}_\Gamma = \text{AIRE}_\Gamma$.

Proof. We have that $\overline{\text{FIRE}}_\Gamma \supseteq \text{AIRE}_\Gamma$ by Proposition 3.10, so we must show that weak limits of FIREs are AIREs.

Let Γ be an exact group and ρ_n a sequence of FIREs weakly converging to ρ . We fix a topologically amenable action $\Gamma \curvearrowright X$ on a compact metric space. We will construct a classwise coherent X -marking of ρ .

Let $\tilde{\rho}_n$ be the classwise coherent X -marking constructed from ρ_n as in Example 4.2. Since $\mathcal{E} \times \text{Prob}(X)$ is sequentially compact, there is a (subsequential) weak limit $\tilde{\rho}$. Then $\tilde{\rho}$ is a classwise coherent X -marking of ρ : it projects onto ρ by weak convergence and continuity of the projection map. Local equivariance follows from the fact that the set of classwise coherent X -markings of ρ is closed. By Proposition 4.3, this shows that ρ is an AIRE. \square

5. NONEXACT GROUPS AND WEAK LIMITS OF FIRES

The aim of this section is to prove a strong converse to Theorem 4.5 by showing that it fails in every nonexact group.

Theorem 5.1. Let Γ be a nonexact countable group. Then there exists a weakly convergent sequence of FIRES in Γ whose limit is not an AIRE.

Let us briefly sketch the main idea of the proof and outline the section. A (finitely generated) nonexact group admits a sequence of small scale expanders. We construct finite IREs such that the cell of the identity is (up to an ε error) one of these small scale expanders with probability at least ε (see Section 5.1). In Section 5.2, we show that small scale expanders are uniformly far from hyperfinite. This then implies, roughly speaking, that the Benjamini–Schramm limit of the connected component of the identity cannot be a hyperfinite unimodular random graph, which in turn implies the limit IRE is not amenable.

In this section we will describe IREs using standard random variable notation from probability theory. Thus, an expression like “let R be an IRE” formally means that R is a measurable map $R : (\Omega, \mathbb{P}) \rightarrow \mathcal{E}$ defined on some implicit probability space. The **distribution** of R is then the pushforward measure $R_*(\mathbb{P})$, which we will typically denote by ρ . If $A \subseteq \mathcal{E}$ is an **event**, then we use the standard shorthand

$$\mathbb{P}[R \in A] := \rho(R^{-1}(A)) = \mathbb{P}[\{\omega \in \Omega \mid R(\omega) \in A\}].$$

5.1. IREs from a sequence of finite subsets. We will now demonstrate that given a finite subset $A \subseteq \Gamma$ we may find a FIRE such that the cell at the identity is a translate of A up to a “small” error with uniformly positive probability.

Denote by $|o|_R$ the size of the class $[e]_R$.

Proposition 5.2. Let Γ be a countable group and $(A_n)_{n \in \mathbb{N}}$ a sequence of finite subsets of Γ . Then, for every $\varepsilon > 0$ there exists a sequence of FIRES $(R_n)_{n \in \mathbb{N}}$ such that

$$\inf_{n \in \mathbb{N}} \mathbb{P}[|o|_{R_n} \geq (1 - \varepsilon)|A_n|] > 0$$

and such that $[o]_{R_n} \subseteq gA_n$ for some $g \in \Gamma$ almost surely.

The proof relies on the following construction of a random partial tiling R of Γ associated to a finite subset $A \subseteq \Gamma$ which was suggested to us by Mikołaj Frączyk after Tom Hutchcroft used it to show that Gromov monsters admit a sequence of finite unimodular random subgraphs whose limit is nonamenable.

Fix a finite subset $e \in A \subseteq \Gamma$ and a parameter $\delta > 0$. Let Π be a Γ -invariant Bernoulli random subset of Γ with an independent uniform $[0, 1]$ mark on each element of Π such that $\mathbb{P}[e \in \Pi] = \frac{\delta}{|A|}$.

The FIRE R is defined by the following sampling process. First, sample from Π . For each $x \in \Pi$, place an A -translate xA . We say that a vertex $v \in \Gamma$ is *conflictive* if there exist

distinct $x, y \in \Pi$ that $v \in xA \cap yA$. The random equivalence relation R is then generated by:

- All vertices in the complement of ΠA belong to singleton classes.
- If $v \in \Pi A$ is not conflictive, then there exists a unique $x_v \in \Pi$ such that $v \in x_v A$, and we let $(v, x_v) \in R$.
- If v is conflictive and not in Π , let $x_v \in \Pi$ be the vertex with the least $[0, 1]$ mark such that $v \in x_v A$, and set $(v, x_v) \in R$.

Equivalently, every $g \in \Pi A - \Pi$ chooses $h_g \in \Pi$ with the smallest $[0, 1]$ mark such that $g \in h_g A$. The remaining $g \in \Gamma$ choose themselves $h_g := g$. The random relation R is then given by the color classes of $g \mapsto h_g$.

The random equivalence relation R clearly has every class contained inside xA for some $x \in \Gamma$ and is invariant as it is constructed as a factor of iid.

In Lemma 5.3, we collect some estimates for R . Let $o \in [\Pi]_R$ denote the event where the identity e is R -related to a vertex in Π and let $o \notin [\Pi]_R$ denote the negation of that event.

Lemma 5.3. The following inequalities hold for R .

- (i) $\mathbb{P}[o \in [\Pi]_R] \geq \delta - \delta^2$,
- (ii) $\mathbb{E}[|o|_R \mid o \in [\Pi]_R] \geq |A|(1 - 2\delta)$, and
- (iii) $\mathbb{P}\left[\frac{|o|_R}{|A|} \geq (1 - 2\delta)^2 \mid o \in [\Pi]_R\right] \geq 4\delta^2(1 - 2\delta)^2$.

Proof. (i) By mass transport and linearity of expectation, we can compute

$$\begin{aligned} \mathbb{P}[o \in [\Pi]_R] &= \mathbb{E}[|o|_R \mathbb{1}_{e \in \Pi}] = |A| \mathbb{P}[e \in \Pi] - \mathbb{E}\left[\sum_{g \in A} \mathbb{1}_{(g, e) \notin R} \mathbb{1}_{e \in \Pi}\right] \\ &= \delta - \sum_{g \in A} \mathbb{P}[(g, e) \notin R \text{ and } e \in \Pi] \end{aligned}$$

leaving us to find upper bounds for the latter probabilities. For $g \in A$,

$$\mathbb{P}[(g, e) \notin R \text{ and } e \in \Pi] \leq \mathbb{P}[g \text{ is conflictive and } e \in \Pi] \quad (5.1)$$

which we can further bound since, for $g \in A$ to be conflictive, the set $gA^{-1} - \{e\}$ must contain some point of Π . Since Π is Bernoulli random, for $g \in A$, we get

$$\begin{aligned} \mathbb{P}[g \text{ is conflictive and } e \in \Pi] &= \frac{\delta}{|A|} \left[1 - \left(1 - \frac{\delta}{|A|}\right)^{|A|-1}\right] \\ &\leq \frac{\delta}{|A|} (|A| - 1) \frac{\delta}{|A|} \leq \frac{\delta^2}{|A|} \quad (\text{Bernoulli inequality}) \end{aligned} \quad (5.2)$$

from which it follows that

$$\mathbb{P}[o \in [\Pi]_R] \geq \delta - \delta^2.$$

(ii) Whenever $o \in [\Pi]_R$, the identity is R -related to a unique element of Π which we denote by x_o . By linearity of expectation and mass transport,

$$\begin{aligned} \mathbb{E}\left[|o|_R \mathbb{1}_{o \in [\Pi]_R}\right] &= |A| \mathbb{P}[o \in [\Pi]_R] - \sum_{g \in A} \mathbb{E}\left[\mathbb{1}_{(x_o, g, x_o) \notin R} \mathbb{1}_{o \in [\Pi]_R}\right] \\ &= |A| \mathbb{P}[o \in [\Pi]_R] - \sum_{g \in A} \mathbb{E}\left[|o|_R \mathbb{1}_{(g, e) \notin R} \mathbb{1}_{e \in \Pi}\right] \\ &\geq |A| \mathbb{P}[o \in [\Pi]_R] - |A| \sum_{g \in A} \mathbb{P}[(g, e) \notin R \text{ and } e \in \Pi] \end{aligned}$$

which implies

$$\mathbb{E}\left[|o|_R \mathbb{1}_{o \in [\Pi]_R}\right] \geq |A| (\delta - \delta^2 - \delta^2)$$

when combined with (i) as well as equations (5.1) and (5.2).

Since $\mathbb{P}[e \in \Pi] = \frac{\delta}{|A|}$ and $|o|_R \leq |A|$ a mass transport argument shows $\mathbb{P}[o \in [\Pi]_R] \leq \delta$. It follows that

$$\mathbb{E}\left[|o|_R \mid o \in [\Pi]_R\right] \geq \frac{1}{\delta} \mathbb{E}\left[|o|_R \mathbb{1}_{o \in [\Pi]_R}\right] \geq |A|(1 - 2\delta).$$

(iii) This is a straightforward application of the Paley–Zygmund inequality to the random variable $\frac{|o|_R}{|A|}$ conditioned on $o \in [\Pi]_R$, using part (ii) and $\mathbb{E}[|o|_R^2 \mid o \in [\Pi]_R] \leq |A|^2$. \square

Proof of Proposition 5.2. Let $\delta > 0$ be small enough so that

$$(1 - 2\delta)^2 \geq 1 - \varepsilon.$$

For each $n \in \mathbb{N}$, let R_n be the FIRE associated to A_n and δ , as constructed in the preamble of the previous lemma. Conclusions (i) and (iii) of Lemma 5.3 imply that

$$\mathbb{P}[|o|_{R_n} \geq (1 - \varepsilon)|A_n|] \geq (\delta - \delta^2)4\delta^2(1 - 2\delta)^2 > 0,$$

proving the proposition. \square

5.2. Small scale expanders are uniformly far from hyperfinite. Small scale expander graphs are non-hyperfinite in a robust way: if one deletes a sufficiently small proportion of the vertices from a small scale expander, the resulting graph cannot be hyperfinite. More precisely:

Proposition 5.4. Let G be a small scale (κ, N) -expander with degree at most $d \in \mathbb{N}$. Let $0 < \varepsilon < \frac{\kappa}{2(1+d)+\kappa}$. Then any induced subgraph $G[A] \subseteq G$ with $|A| \geq (1 - \varepsilon)|V(G)|$ is not (ε, N) -hyperfinite.

Proof. We will prove the contrapositive. Suppose G is a small scale (κ, N) -expander with degree at most d and that there exists $A \subseteq V(G)$ with $|A| \geq (1 - \varepsilon)|V(G)|$ such that $G[A]$ is (ε, N) -hyperfinite. Hence there exists a set $E \subseteq E(G[A])$ such that $|E| \leq \varepsilon|A|$ and $G[A] - E$ has connected components of size at most N .

Let \mathcal{F} be the family of connected components of $G[A] - E$, and $E(\mathcal{F})$ denote the union of the edge sets of the graphs of \mathcal{F} . Thus,

$$|E(G) - E(\mathcal{F})| \leq |E| + d|V(G) - A| \leq \varepsilon|A| + \varepsilon d|V(G)| \leq \varepsilon(1 + d)|V(G)|.$$

On the other hand, we get the following bound from small scale expansion,

$$|E(G) - E(\mathcal{F})| \geq \frac{1}{2} \sum_{F \in \mathcal{F}} |\partial_G F| \geq \frac{\kappa}{2} \sum_{F \in \mathcal{F}} |F| = \frac{\kappa}{2} |A| \geq \frac{\kappa}{2} (1 - \varepsilon) |V(G)|$$

where the first inequality is true because the sum $\sum_{F \in \mathcal{F}} |\partial_G F|$ counts edges between \mathcal{F} components at most twice, and edges from a \mathcal{F} component to $V(G) - A$ at most once.

We have shown that $\kappa(1 - \varepsilon) \leq 2\varepsilon(1 + d)$ and thus $\frac{\kappa}{2(1+d)+\kappa} \leq \varepsilon$, concluding the proof. \square

5.3. Nonamenable weak limits of FIREs. Recall if \mathcal{C} is a Cayley graph of Γ and R is an IRE, we let $\mathcal{C}(o, R) := \mathcal{C}[[e]_R]$ denote the (possibly disconnected) random subgraph of \mathcal{C} induced by $[e]_R$ and rooted at $o = e$. We then distinguish this from $\mathcal{C}_{\text{con}}(o, R)$ which will denote the connected component of o in $\mathcal{C}(o, R)$.

Before proving the main theorem, let us introduce the coinduction construction. For an inclusion of groups $\Lambda \leq \Gamma$ and an IRE R on Λ , define the coinduced IRE $\text{CInd}_\Lambda^\Gamma(R)$ by sampling an independent copy of R on every left coset of Λ . The following properties are immediate from the definition.

Proposition 5.5. Let $\Lambda \leq \Gamma$ be an inclusion of groups and R an IRE on Λ . Then:

- (1) $\text{CInd}_\Lambda^\Gamma(R)$ is an IRE on Γ .
- (2) R is amenable (resp. finite) if and only if $\text{CInd}_\Lambda^\Gamma(R)$ is amenable (resp. finite).
- (3) The map $\text{CInd}_\Lambda^\Gamma$ is continuous with respect to the weak topology.

We can now prove that nonexact countable groups admit a convergent sequence of finite IREs whose limit is a nonamenable IRE.

Proof of Theorem 5.1. We first reduce to the case where Γ is finitely generated.

Since increasing unions of exact groups are exact [BO08, Exercise 5.1.1], if Γ is nonexact, there is a finitely generated subgroup $\Lambda \leq \Gamma$ that is also nonexact. Suppose that $(R_n)_n$ is a sequence of finite IREs on Λ weakly converging to a nonamenable IRE R on Λ . Proposition 5.5 then implies that the sequence $\text{CInd}_\Lambda^\Gamma(R_n)$ of finite IREs converges weakly to the nonamenable IRE $\text{CInd}_\Lambda^\Gamma(R)$.

Now suppose Γ is finitely generated nonexact and let \mathcal{C} be a Cayley graph for Γ whose degree we denote by d . Then there exists a sequence $(G_n)_n$ of induced subgraphs of \mathcal{C} with $e \in V(G_n)$ that forms a sequence of small scale κ -expanders, for some fixed $\kappa > 0$. Fix $\varepsilon > 0$ where ε is small enough to satisfy the hypotheses of Proposition 5.4 and let $(R_n)_n$ be the sequence of FIREs obtained by applying Proposition 5.2 with inputs ε and the sequence of subsets $(V(G_n))_n$. By sequential compactness we may assume, upon possibly passing to a

subsequence, that the sequence $(R_n)_n$ of FIREs weakly converges to an IRE R . It remains to show that R is nonamenable.

Let us therefore suppose for the sake of contradiction that R is an amenable IRE. We will find $N \in \mathbb{N}$ and a subset $A \subseteq V(G_N)$ with $|A| \geq (1 - \varepsilon)|V(G)|$ such that the induced subgraph $G_N[A]$ is (ε, N) -hyperfinite, contradicting Theorem 5.4.

A coupling argument using [CFW81] as in Proposition 3.10 shows that $\mathcal{C}_{\text{con}}(o, R)$ may be obtained by sampling connected components of a μ -hyperfinite Borel graph. Hence $\mathcal{C}_{\text{con}}(o, R)$ is a hyperfinite unimodular random graph by Proposition 2.9. Let $\delta > 0$ be small enough such that

$$\frac{\delta}{2c(1 - \varepsilon)} \leq \varepsilon,$$

where $c := \inf_{n \in \mathbb{N}} \mathbb{P}[|o|_{R_n} \geq (1 - \varepsilon)|V(G_n)|] > 0$. It follows from Proposition 3.5 and Theorem 2.10 that there exists $k \in \mathbb{N}$ such that for n sufficiently large $\mathcal{C}_{\text{con}}(o, R_n)$ is (δ, k) -hyperfinite. Hence there exists N such that $\mathcal{C}_{\text{con}}(o, R_N)$ is (δ, N) -hyperfinite. Thus there exists a rerooting equivariant subset of edges $K(\mathcal{C}_{\text{con}}(o, R_N)) \subset E(\mathcal{C}_{\text{con}}(o, R_N))$ such that $\mathbb{E}[\deg_{K(\mathcal{C}_{\text{con}}(o, R_N))}(o)] \leq \delta$ and $\mathcal{C}_{\text{con}}(o, R_N) - K(\mathcal{C}_{\text{con}}(o, R_N))$ has connected components of size at most k almost surely.

Define a coupling $(\mathcal{C}(o, R_N), E_N, o)$ by letting

$$E_N := \bigcup_{g \in [e]_{R_N}} gK(\mathcal{C}(o, g^{-1}.R_N)) \subset E(\mathcal{C}(o, R_N)).$$

By construction E_N coincides with $K(\mathcal{C}_{\text{con}}(o, R_N))$ on each connected component of $\mathcal{C}(o, R_N)$. Therefore $\mathbb{E}[\deg_{E_N}(o)] \leq \delta$ and $\mathcal{C}(o, R_N) - E_N$ has connected components of size at most k almost surely.

We can then compute

$$\begin{aligned} & \mathbb{E}\left[\frac{|E_N|}{|o|_{R_N}} \mid |o|_{R_N} \geq (1 - \varepsilon)|V(G_N)|\right] \\ & \leq \frac{1}{(1 - \varepsilon)|V(G_N)|} \mathbb{E}\left[\frac{1}{2} \sum_{v \in V(\mathcal{C}(o, R_N))} \deg_{E_N}(v) \mid |o|_{R_N} \geq (1 - \varepsilon)|V(G_N)|\right] \\ & \leq \frac{1}{2(1 - \varepsilon)} \mathbb{E}\left[\deg_{E_N}(o) \mid |o|_{R_N} \geq (1 - \varepsilon)|V(G_N)|\right] \end{aligned}$$

by the conditioning assumption and mass transport. A further upper bound

$$\mathbb{E}\left[\frac{|E_N|}{|o|_{R_N}} \mid |o|_{R_N} \geq (1 - \varepsilon)|V(G_N)|\right] \leq \frac{1}{2(1 - \varepsilon)} \frac{\mathbb{E}[\deg_{E_N}(o)]}{\mathbb{P}[|o|_{R_N} \geq (1 - \varepsilon)|V(G_N)|]} \leq \frac{\delta}{2c(1 - \varepsilon)} \leq \varepsilon$$

is established by using the general bound $\mathbb{E}[X \mid A] \leq \mathbb{E}[X]/\mathbb{P}[A]$ for non-negative random variables. It then follows from the first moment principle that there exists a finite subset $A \subseteq V(G_N)$ with $|A| \geq (1 - \varepsilon)|V(G_N)|$ such that $G[A]$ is (ε, N) -hyperfinite, contradicting Theorem 5.4 and showing the limit IRE R is not amenable. \square

6. IDEAL BERNOULLI VORONOI TESSELLATIONS AND QUESTIONS

This paper grew out of a desire to better understand ideal Bernoulli Voronoi tessellations, the discrete analog of ideal Poisson Voronoi tessellations. The ideal Bernoulli Voronoi tessellation was first studied in the case of regular trees by Bhupatiraju [Bhu10].

Fix a left-invariant proper metric d on Γ (for example, the word metric associated to a finite generating set). The **Voronoi tessellation** associated to a configuration $\omega \subseteq \Gamma$ is the ensemble of sets $\{V_\omega(g)\}_{g \in \Gamma}$, where

$$V_\omega(g) := \{x \in \Gamma \mid d(x, g) \leq d(x, h) \text{ for all } h \in \omega\}.$$

Note that the Voronoi tessellation is equivariantly defined (as the metric is left-invariant). It is not a genuine partition however, as the cells can have boundary points in common. It is easy to refine the Voronoi tessellation to a genuine partition whilst maintaining equivariance: for example, each point on the boundary of a cell (that is, whose minimal distance to ω is achieved by multiple points of ω) simply picks one at random. There are other methods to resolve the boundaries, we fix one and continue to use the above notation for the refined version.

The **Bernoulli Voronoi tessellation** BVT_p with parameter $p \in (0, 1)$ is the Voronoi tessellation of the Bernoulli random subset $\Pi_p \subseteq \Gamma$ (that is, each $\gamma \in \Gamma$ is independently in Π_p with probability p). BVT_p is an example of a finite IRE as implied by the following lemma.

Lemma 6.1. Let Π be a Γ -invariant random subset of Γ with intensity $\mathbb{P}[e \in \Gamma] > 0$. Suppose $C(g, \Pi) \subseteq \Gamma$ is a family of measurably defined subsets for each $g \in \Pi$ such that

- (Equivariance) $C(\gamma g, \gamma \Pi) = \gamma C(g, \Pi)$ for all $\gamma \in \Gamma$, and
- (Disjointness) $C(g, \Pi) \cap C(h, \Pi) = \emptyset$ for all distinct $g, h \in \Pi$.

Then Π almost surely, every cell $C(g, \Pi)$ is finite.

Proof. We get that $\mathbb{E}[|C(e, \Pi)| \mathbb{1}_{e \in \Pi}] \leq 1$ by applying mass transport with the function $f(x, y; \Pi) = \mathbb{1}_{x \in C(y, \Pi)}$. It follows that $|C(e, \Pi)| \mathbb{1}_{e \in \Pi}$ is finite almost surely, and therefore $\mathbb{P}[g \in \Pi \text{ and } C(g, \Pi) \text{ is infinite}] = 0$ for each $g \in \Gamma$ by invariance.

Since Γ is countable, almost surely for every $g \in \Gamma$ either $g \notin \Pi$ or the cell $C(g, \Pi)$ is finite. □

For a nonamenable group Γ , any subsequential weak limit of BVT_p defines a nontrivial IRE on Γ . Indeed, the nontriviality follows from Kechris [Kec17, Theorem 15.7]. We call these subsequential weak limits *ideal Bernoulli Voronoi tessellations*. Let us state some basic questions about them.

One step in the proof of fixed price for higher rank semisimple Lie groups [FMW23] consists of proving that the ideal Poisson Voronoi tessellation is an amenable IRE. It is natural to raise the question of amenability in general.

Question 1. Are ideal Bernoulli Voronoi tessellations amenable as IREs?

It follows from Theorem 4.5 that for exact groups the answer is positive. However, Theorem 5.1 shows that it is not sufficient to be a weak limit of FIREs in order to be amenable and so the general question remains open.

In the continuous setting, it is known that there is a unique limit for the ideal Poisson Voronoi tessellation on hyperbolic space [DCE⁺23] and more general symmetric spaces [FMW23]. Bhupatiraju [Bhu10] showed that the limit is unique for regular trees.

Question 2. In what generality can one say that there is a *unique* ideal Bernoulli Voronoi tessellation on a group?

We may view \mathcal{E} as a subset of $(2^\Gamma)^\Gamma$ by identifying each $r \in \mathcal{E}$ with the function $\gamma \mapsto [\gamma]_r$. A **cell selection rule** is a measurable and equivariant map $\varphi : \mathcal{E} \rightarrow (2^\Gamma)^\Gamma$ such that for all $\gamma \in \Gamma$, either $\varphi_r(\gamma) = [\gamma]_r$ or $\varphi_r(\gamma) = \emptyset$. That is, φ is a measurable way of selecting some of the cells of equivalence relations. A cell selection rule is **trivial** if it selects all cells, or selects no cells. An IRE ρ is **indistinguishable** if it admits no nontrivial cell selection rules. The second author [Mel24] recently showed that the ideal Poisson Voronoi tessellation on symmetric spaces is indistinguishable.

Question 3. Are the ideal Bernoulli Voronoi tessellations always indistinguishable? Under what conditions is a weak limit of finite IREs indistinguishable?

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