Complexity of the isomorphism of Archimedean orders on $\mathbb{Z}^2$

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March, 2022

A thesis submitted to McGill University in partial fulfillment of the requirements of the degree of

Master of Mathematics

Abstract

In this thesis, we present the theory of left-orderable groups as well as the theory of Borel complexity. Motivated by a question of F. Calderoni, D. Marker, L. Motto Ros and A. Shani, we prove the following result: the isomorphism relation on Archimedean orders of $\mathbb{Z}^2$ is hyperfinite, but not smooth.

Résumé

Dans ce mémoire, nous présentons la théorie des groupes ordonnables et la théorie de la complexité borélienne. Motivé par une question de F. Calderoni, D. Marker, L. Motto Ros et A. Shani, nous prouvons le théorème suivant: la relation d'isomorphisme sur les ordres Archimédien de $\mathbb{Z}^2$ est hyperfinie, mais n'est pas concrètement classifiable.

Acknowledgement

I wish to express sincere thanks to my supervisor Marcin Sabok for immesurable help. I also would like to thank Filippo Calderoni for interesting discussion about his work, as well as Denali Reyes and Hugues Bellemare for always being there to listen to me ramble about Borel complexity.

Contents

1 Introduction

The theory of Borel complexity has a rich range of applications in classification problems. This theory defines a powerful way to study relative complexity of isomorphism problems and more generally equivalence problems, especially those arising from group actions. To motivate the introduction of Borel complexity, let us consider a naïve example:

Example 1.1. Take the class of finitely generated abelian groups and its isomorphism relation. By the finitely generated abelian group structure theorem, if G is a finitely generated abelian group, there is a unique decomposition of the form

$$G \cong \mathbb{Z}/p_1^{r_1}\mathbb{Z} \times ... \times \mathbb{Z}/p_n^{r_n} \times \mathbb{Z}^r,$$

where $p_1 \leq ... \leq p_n$ and if $p_i = p_{i+k}$ with $k \in \mathbb{N}$, then $r_i \leq r_{i+k}$. This theorem allows us to associate to each isomorphism class a well-defined sequence $(r, p_1, r_1, ..., p_n, r_n, 0, 0, ...)$ which can be encoded in the decimal representation of a real number r_G. We call this number a numerical invariant for G. These numerical invariants are fairly easy to describe.
Another observation is that the structure theorem gives a canonical representative for every isomorphism class. These are again simple to describe. These two properties make this example a simple equivalence relation.

Of course, there is no mathematical rigor to the use of the words "simple" or "canonical" in the preceding example. Let us study another example:

Example 1.2. Consider the interval $[0, 1]$ and declare that two numbers $r, s \in [0, 1]$ are equivalent if $r - s \in \mathbb{Q}$. Here, a set of representatives is called a Vitali set. Vitali sets are known to be non-measurable and were the first example of such sets. This makes any set of representatives for each class very complex and unsatisfying.
Can this relation have simple numerical invariants? In sense to be defined later, it is impossible, since canonical representatives and simple numerical invariants are linked.

It is well known that the axiom of choice gives us a set of representatives for any equivalence relation, but such representatives are often impossible to describe, let alone construct. This is the case of the second example. Descriptive Set Theory tries to avoid the more annoying consequences of the axiom of choice and gives a class of sets that we are able to describe.

We will formalize all of this in Chapter 2, where we will define a notion of simple equivalence relations, called "smooth". In this case, the first example is smooth while the second isn't. However, it does satisfy another condition called "hyperfiniteness", which guarantees that it is not too complicated.

Our goal here is to apply Borel complexity to left-orderable groups. Invariant ordering on groups have been studied extensively, going as far back as the work of Dedekind, Hölder and Hilbert in the 19th century. One way in which the theory is rich is the range of mathematical tools which can be applied to it. Other than abstract algebra, dynamics and topology are fruitful ways to think of left-orderable groups.

A classic dynamical result on left-orderable groups states that a group acting faithfully in an order-preserving way on the real line must be left-orderable. This is the standard dynamical technique of studying left-orderable groups by their actions on ordered sets. See Andrés Navas' article [8] for a presentation of dynamical methods associated to left-orderable groups.

A particularly interesting topological conjecture is the L-space conjecture, which asks whether there is an equivalence between left-orderability of the fundamental group of a 3-manifold (with some additional properties) and the existence of a foliation with desirable properties on said manifold. This has already been established for manifolds admitting a Seifert Fibered structure, that is those admitting a fibration over a circle with some regularity properties. A good introduction to left-orderable groups and their link to low-dimensional topology can be found in [3] by Adam Clay and Dale Rolfsen.

Recently, many authors have studied left-orderable groups from the point of view of Borel complexity. The automorphisms of a group G give rise to a natural action on the space of left-orders, which gives a relation we call the isomorphism relation on left-orders. One can also restrict the action to the inner automorphisms of G, which gives the conjugacy action of G on the space of left-orders on the group.

Adam Clay and Filippo Calderoni have proven in [1] that for a large class of left-orderable groups G, the following is true: the conjugacy action of G on the space of its left-orders induces a universal countable Borel equivalence relation. In particular, this class contains the free groups, pure braid groups and is closed under both direct and free products.

On the other hand, F. Calderoni, D. Marker, L. Motto Ros and A. Shani have studied in [2] the isomorphism relation on more specific class of orderings, called Archimedean. Intuitively, Archimedean groups are those without any element which is infinitely small with respect to another element. One of their theorems is the following:

Theorem 1.3 ([2])**.** *The isomorphism relation on Archimedean orderings of $\mathbb{Q}^2$ is not smooth. In particular, isomorphism of Archimedean-ordered group is not smooth.*

In [2], F. Calderoni, D. Marker, L. Motto Ros and A. Shani also ask the following question:

Question 1.4 ([2], Question 2.8)**.** *Is the isomorphism relation on Archimedean orderings of $\mathbb{Q}^2$ hyperfinite?*

The same question can be asked for a more elementary group, namely $\mathbb{Z}^2$ in place of $\mathbb{Q}^2$. Our main result is as follows:

Theorem 1.5. *The isomorphism relation on Archimedean orderings of $\mathbb{Z}^2$ is hyperfinite, but not smooth.*

In particular, Theorem 1.5 implies that the isomorphism relation on all Archimedean-ordered finitely generated groups is not smooth.

2 Borel complexity

In this section, we provide the necessary background and present the theory of invariants we alluded to in the introduction.

Definition 2.1. A topological space X is said to be **Polish** if it is separable and there exists a complete metric generating its topology.

Examples 2.2.

- $\mathbb{R}$ with the usual metric is a complete space with $\mathbb{Q}$ dense in it, hence is Polish.
- Separable Banach spaces are Polish.
- $(0,1)$ is not closed in $\mathbb{R}$ and so is not complete with the induced metric. It is however a G_δ subset of $\mathbb{R}$, that is a countable union of closed sets. In general, we have the following proposition:

Proposition 2.3. *If X is a Polish space and $Y \subset X$, Y is a Polish space under the induced topology if and only if Y is a G_δ in X.*

- The Cantor space $2^{\mathbb{N}}$ and Baire space $\mathbb{N}^{\mathbb{N}}$ can both be given a complete metric defined by

$$\log d(x,y) = -\min\{i : x(i) \neq y(i)\}.$$

  This metric generates the product topology, which is separable. Hence, both $2^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$ are Polish.

- Countable products of Polish spaces are Polish.

One interesting fact about Polish spaces is that they respect the Continuum Hypothesis, in the sense of the following proposition:

Proposition 2.4. *If X is an uncountable Polish space, X is in bijection with $\mathbb{R}$.*

Sketch of proof. To prove this proposition, one splits the point of X by whether they have a countable neighbourhood or not. Only countably many points can have this property, due to separability and metrizability of X. By removing these countably many points, one is left with a subspace we call the perfect kernel of X, which has no isolated points.

From a perfect Polish space X_0, that is one without isolated points, one can construct an injection from $2^{\mathbb{N}} \hookrightarrow X_0$ using machinery called Cantor schemes. Conversely, one can inject X^0 into the Cantor space $2^{\mathbb{N}}$ by fixing a countable sequence of open sets generating the topology and encoding points of X using it. Such a sequence can be found by fixing a metric and a countable set witnessing separability. ■

Polish spaces have a rich structure when paired with the following definition:

Definition 2.5. Given a Polish space X, we construct the **Borel σ-algebra** as the smallest collection $\mathcal{B}_X$ of subsets of X such that:

- $\mathcal{B}_X$ contains every open and closet set of X.
- $\mathcal{B}_X$ is closed under complements: If $Y \in \mathcal{B}_X$, $Y^c \in \mathcal{B}_X$.
- $\mathcal{B}_X$ is closed under countable union and intersection: If $Y_1, Y_2, ...$ is a sequence in $\mathcal{B}_X$, $\bigcup_i Y_i \in \mathcal{B}_X$ and $\bigcap_i Y_i \in \mathcal{B}_X$.

If $Y \in \mathcal{B}_X$, we simply say that Y is Borel. In case the underlying space is clear, we only note $\mathcal{B}$.

Definition 2.6. If X, Y are two Polish spaces and $f : X \to Y$ is a function between them, we say that f is **Borel** if

$$\forall B \in \mathcal{B}_Y, f^{-1}(B) \in \mathcal{B}_X$$

In usual measure theoretic language, this says that f is measurable with respect to the Borel σ-algebra of each space.

We have the following standard theorem which completely anwers the question of isomorphism of Polish spaces and their σ-algebra:

Theorem 2.7 (Borel Isomorphim Theorem)**.** *If X, Y are two Polish spaces of the same cardinality, they are Borel-isomorphic in the following sense: there exists a bijection $f : X \to Y$ such that both f, f^{-1} are Borel.*

This theorem is a considerable strenghtening of Proposition 2.4. It also motivates the following definition:

Definition 2.8. If X is a set along with a σ-algebra $\mathcal{A}$, we say that X is a **standard Borel space** if it there is a σ-algebra isomorphism between $(X, \mathcal{A})$ and $(Y, \mathcal{B}_Y)$, where Y is some Polish space. Standard Borel spaces are, up to Borel isomorphism, uniquely defined by their cardinality, by the Borel Isomorphism Theorem.

Examples 2.9.

- If $n \in \{1, 2, ..., \mathbb{N}\}$, the only standard Borel structure on a space X with $|X| = n$ is the discrete one. We refer to countable spaces directly by their cardinality, that is we denote n for the standard Borel space with n elements.
- $\mathbb{R}$ and $(0, 1)$ are isomorphic standard Borel spaces. In fact, they are homeomorphic, which is much stronger. However, we have no such homeomorphism between $\mathbb{R}$ and $[0, 1]$, yet they are Borel isomorphic.

2.1 Borel equivalence relations

Since Borel sets are, informally, those we can describe with countable information from the topology, we wish to base the desired theory of invariants on the notion of Borel subsets and functions.

Definition 2.10. An equivalence relation E on a standard Borel space X is said to be a **Borel equivalence relation** if it is Borel as a subset of X^2. In the case where every equivalence class is countable (resp. finite), we say that E is a countable (resp. finite) Borel equivalence relation.

Definition 2.11. Let X, Y be two Polish spaces with respective Borel equivalence relations E_X, E_Y. A Borel function $f : X \to Y$ is said to be a **reduction** from E_X to E_Y if

$$x \, E_X \, x' \iff f(x) \, E_Y \, f(x').$$

If the equivalence relations are clear, we just say that f is a reduction.

Remark 2.12. The previous definition is of little use without the Borel condition. Using choice, the notion of non-Borel reduction only compares the number of equivalence classes of E_X and E_Y. One can also consider slightly bigger σ-algebras, such as those of Baire-measurable sets or μ-measurable sets, where μ is a Borel probability measure on X. Both of these amount to adding "negligible" sets, either from the measure or topology point of view.

We give some properties of reductions:

Proposition 2.13. *Let X, Y, Z be Polish spaces with respective Borel equivalence relations E_X, E_Y, E_Z and Borel functions $f : X \to Y$, $g : Y \to Z$.*

- *If both f, g are reductions, $g \circ f : X \to Z$ is also a reduction.*
- *Let G be a countable group with two Borel actions $G \curvearrowright X$ and $G \curvearrowright Y$ such that $E_X = E_G^X$ and $E_Y = E_G^Y$. If f is G-invariant and the pre-image of every point is contained in a single E_G^X-class, then f is a reduction. That is,*

$$\begin{gathered}\forall x \in X, \, \forall g \in G, \, f(g \cdot x) = g \cdot f(x) \text{ and} \\ \big(\forall x, x' \in X, \, f(x) = f(x') \Longrightarrow x \, E_G^X \, x'\big) \Longrightarrow f \text{ is a reduction.}\end{gathered}$$

Proof. For the first property, consider the equivalence

$$x \, E_X \, x' \iff f(x) \, E_Y \, f(x') \iff g(f(x)) \, E_Z \, g(f(x')).$$

For the second property, using G-invariance of f:

$$\begin{aligned} x \, E_G^X \, x' &\Longrightarrow \exists g \in G, \, g \cdot x = x' \\ &\Longrightarrow \exists g \in G, \, g \cdot f(x) = f(x') \\ &\Longrightarrow f(x) \, E_G^Y \, f(x'). \end{aligned}$$

For the reverse implication, we use both G-invariance of f and the fact that pre-images are E_G^X-invariant:

$$\begin{aligned} f(x) \, E_G^Y \, f(x') &\Longrightarrow \exists g \in G, \, g \cdot f(x) = f(x') \\ &\Longrightarrow \exists g \in G, \, f(g \cdot x) = f(x') \\ &\Longrightarrow \exists g \in G, \, (g \cdot x) \, E_G^X \, x' \\ &\Longrightarrow x \, E_G^X \, x'. \end{aligned}$$

The last implication uses the fact that $\forall g \in G$, $x \, E_G^X \, (g \cdot x)$. ■

Actions from countable groups are a great way to introduce countable Borel equivalence relations.

Definition 2.14. Given a countable group G and a Borel action $G \curvearrowright X$, we define the **orbit equivalence relation**

$$x \, E_G^X \, x' \iff \exists g \in G, g \cdot x = x'.$$

This is a countable Borel equivalence relation.

Let us go through a few examples of useful Borel equivalence relations:

Examples 2.15.

- If X is a Polish space, its diagonal is a closed subset of X^2, since X is Hausdorff. Thus, the identity relation, which we denote $=_X$, is a finite Borel equivalence relation.

- If G is a countable group, letting $G \curvearrowright 2^G$ by shifting sequences yields a wide class of Borel equivalence relations. More precisely, if $x : G \to 2$ and $\gamma \in G$,

$$(\gamma \cdot x)\,(g) := x(g \cdot g_1 \cdot g_2).$$

- We define E_0 to be the relation of eventual equality on $2^{\mathbb{N}}$:

$$(x_n) \ E_0 \ (y_n) \iff \exists N, \forall n \geq N, x_n = y_n.$$

  This relation will be of particular interest in the next section.

- If $X = \mathbb{R}$, we consider the action by translations $\mathbb{Q} \curvearrowright \mathbb{R}$ and its orbit equivalence relation E_v'. We call the restriction of E_v' to $[0,1]$ the Vitali equivalence relation, which we denote E_v. This is the second example of the introduction.

Examples 2.16.

- If X is countable, its Borel σ-algebra is discrete, making every function with X as domain Borel. Thus, there is a reduction from $=_X$ to $=_Y$ exactly when $|X| \leq |Y|$.

- The eventual equality relation, E_0, cannot be reduced to $=_{\mathbb{R}}$. We will mention how to prove this in Proposition 2.21.

- The Vitali equivalence relation E_v cannot be reduced to $=_{\mathbb{R}}$. This is as we mentioned in the introduction, when we said that E_v had no suitable numerical invariants. While we could prove this separately from the previous example, if we can show that E_0 reduces to E_v, from composition it is clear that E_v cannot reduce to $=_{\mathbb{R}}$. We can do something stronger, which requires the following definition.

Definition 2.17. Let X, Y be Polish spaces with corresponding Borel equivalence relations E_X, E_Y. If there are reductions $f : X \to Y$ and $g : Y \to X$, we say that E_X and E_Y are **bireducible**.

Examples 2.18.

- For countable Polish spaces, bireducibility of Borel equivalence relations coincide with equality of the cardinality of equivalence classes.

- As hinted at before, we can show that E_0 and E_v are bireducible. This will be done in Corollary 2.28.

2.2 Notable complexities of countable Borel equivalence relations

In this section, we want to give a semi-complete picture of the hierarchy of complexity in countable Borel equivalence relations. Throughout this section, let X/E_X denote the set of equivalence classes of E_X.

In Examples 2.16, we mentioned that if X is a countable space, there is a reduction from $=_X$ to $=_Y$ exactly when $|X| \leq |Y|$. Similarly, if a countable Borel equivalence relation E_X has only countably many equivalence classes, it is bireducible with $=_n$, where $n = |X/ E_X| \in \{1, 2, .., \mathbb{N}\}$.

This shows that $=_1, =_2, ..., =_\mathbb{N}$ forms an initial sequence for the complexity hierarchy. The following theorem strengthens this observation to include $=_\mathbb{R}$:

Theorem 2.19 (Mycielski, Silver). *If X is an uncountable standard Borel space and E_X is a countable Borel equivalence relation, there is a Borel reduction from $=_\mathbb{R}$ to E_X.*

This is a special case of a theorem by Silver [9]. This also follows from an earlier theorem by Mycielski. This theorem motivates the following definition:

Definition 2.20. Given a Polish space X along with a Borel equivalence relation E_X, we say that E_X is **smooth** if it is bireducible to $=_Y$, where Y is some Polish space.

One might ask whether every Borel equivalence relation is smooth, which we can answer by the following proposition:

Proposition 2.21. *E_0 is not smooth.*

Sketch of proof. To prove this, one uses that fact that E_0 is an ergodic equivalence relation: if $A \subset [0, 1]$ is an E_0-invariant Borel set, its Lebesgue measure must be 0 or 1.

Considering a Borel reduction f from E_0 to $=_{2^\mathbb{N}}$, one can split $2^\mathbb{N}$ into points x such that $f(x)(0) = 0$ and points x such that $f(x)(0) = 1$. Since these are two Borel E_0-invariant subsets, one must have full measure. Then, one splits this set again considering what is $f(x)(1)$.

Doing this infinitely many times gives a sequence in $2^\mathbb{N}$ whose preimage needs to be of full measure, but is also a single equivalence class, hence countable. This is a contradiction since the measure on $2^\mathbb{N}$ is non-atomic. ∎

Silver's theorem makes smooth equivalence relations the least elements of the partial order of Borel complexity, with $=_\mathbb{R}$ being the minimum for uncountable spaces. We have that E_0 is a successor to smoothness, given by the following dichotomy.

Theorem 2.22 (Glimm-Effros Dichotomy, [6]). *If X is a standard Borel space with a Borel equivalence relation E_X, exactly one of the following is true:*

- *E_X is smooth,*
- *there is a Borel reduction from E_0 to E_X.*

Let us study E_0 more carefully. One of its most interesting properties is that it is approximable by finite equivalence relation, in the sense of the following proposition:

Proposition 2.23. *There exists an increasing sequence of finite Borel equivalence relations F_i such that $E_0 = \bigcup_{i\in\mathbb{N}} F_i$.*

Proof. Recall that two sequences $x = (x_i), y = (y_i)$ are E_0-equivalent exactly when

$$\exists N \in \mathbb{N}, \forall n \geq N, x_n = y_n.$$

Define F_i as follows:

$$x\ F_i\ y \iff \forall n \geq i, x_n = y_n.$$

Then, F_i has equivalence classes with exactly 2^i elements, implying that they are finite Borel equivalence relations. Further, if $i < j$, then $x\ F_i\ y \Longrightarrow x\ F_j\ y$ and if $x\ E_0\ y$, there is some N such that $x\ F_N\ y$, as required. ∎

Definition 2.24. Given E_X an equivalence relation on a Polish space X, we say that E_X is **hyperfinite** if there exists an increasing sequence of finite Borel equivalence relations $F_i, i \in \mathbb{N}$ such that

$$E = \bigcup_{i\in\mathbb{N}} F_i.$$

Note that every hyperfinite relation is a countable Borel equivalence relation.

Examples 2.25.

- E_0 is hyperfinite, by Proposition 2.23.
- E_v is hyperfinite. To see this, consider the equivalence relations

  $$x\ F_i\ y \iff (x - y)i! \in \mathbb{Z}.$$

  There are at most $i!$ elements in each F_i-classes. These are increasing and their union is the Vitali equivalence relation.

E_0 is in fact the most complex hyperfinite relation, in the sense of the following proposition:

Theorem 2.26 (Dougherty-Jackson-Kechris, [4])**.** *If E_X is hyperfinite, there is a Borel reduction from E_X to E_0.*

Corollary 2.27. *If E_X is hyperfinite, E_X is either smooth or bireducible to E_0.*

Proof. Assume that E_X is not smooth. By the Glimm-Effros Dichotomy, E_0 reduces to E_X. Since E_X is hyperfinite, there is a reduction from E_X to E_0. Hence, E_0 and E_X are bireducible. ∎

Corollary 2.28. *The Vitali equivalence relation E_v is bireducible with E_0. In particular, E_v is not smooth.*

3 Left-orders and Archimedean orders

In this section, we assume that groups are non-trivial.

Definition 3.1. A total order $<$ on a group G is said to be a **left-order** if it is left-invariant, that is if

$$\forall g, h, k \in G,\ g < h \Longleftrightarrow kg < kh.$$

Examples 3.2.

- With the usual orders, $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are all left-orderable.
- If G, H are left-orderable with orders $<_G, <_H$, $G \times H$ is left-orderable with respect to the lexicographic order.
- In particular, $\mathbb{Z}^n, \mathbb{Q}^n, \mathbb{R}^n$ are all left-orderable.
- Given an exact sequence

  $$1 \to H \xrightarrow{\iota} G \xrightarrow{\pi} K \to 1$$

  where H, K are left-orderable with orders $<_H, <_K$, there exists an order on G compatible with $<_K$ and $<_H$, defined by

  $$g_1 <_G g_2 \Longleftrightarrow \begin{cases} \mathrm{id}_K <_K \iota^{-1}(g_1^{-1}g_2) & \text{if } \pi(g_1) = \pi(g_2) \\ \pi(g_1) < \pi(g_2) & \text{else.} \end{cases}$$

- In particular, a semi-direct product of orderable groups is itself orderable.
- The group $G = \langle x, y \,|\, yxy^{-1} = x^{-1} \rangle$ can be described as a semi-direct product $\mathbb{Z} \rtimes \mathbb{Z}$. It follows that it is left-orderable.
- In general, if a group G has a left-order $<$ and $g \in P$ such that $g > \mathrm{id}$,

  $$\mathrm{id} < g < g^2 < g^3 < \ldots$$

  This is impossible if G has torsion. In particular, finite groups are not left-orderable.

The following gives a nice algebraic characterization of left-orderability

Proposition 3.3. *If $<$ is a left-order on G, the set of positive elements of G with respect to $<$,*

$$P_< := \{g \in G : \mathrm{id} < g\}$$

satisfies the two following conditions:

1. *$P_<$ is a sub-semigroup of G.*
2. *G can be partitioned as $G = P_< \sqcup \{\mathrm{id}\} \sqcup P_<^{-1}$.*

Conversely, if P satisfies conditions 1 and 2, P determines a left-order $<_P$ as follows:

$$g <_P h \iff h^{-1}g \in P.$$

Furthermore, the constructions are dual in the following sense:

$$\begin{aligned} P &= \{g \in G : id <_P g\} \\ g < h &\iff h^{-1}g \in P_<. \end{aligned}$$

Definition 3.4. If P satifies the conditions enumerated in Proposition 3.3, we say that it is a **positive cone** of G. In light of that proposition, the **space of left-order** is defined as

$$\mathrm{LO}(G) := \left\{P \in 2^G : P \text{ is a positive cone}\right\}.$$

Since it is defined by closure properties, we can prove that $\mathrm{LO}(G)$ is a closed subset, thus compact as well, hence Polish. If it is still unclear how to prove such a thing, a similar proof is given in Proposition 3.9.

Remark 3.5. If P is a positive cone, P^{-1}, the pointwise inversion of P, is also a positive cone whose order corresponds to reversing that of P. We also refer to the set P^{-1} as the negative elements of $<_P$, since

$$P^{-1} = \{g \in G : \mathrm{id} >_P g\}.$$

We call P, P^{-1} opposite cones.

Definition 3.6. A left-order $<$ on G is said to be **Archimedean** if

$$\forall g, h > \mathrm{id}, \exists n, g^n > h.$$

If P is a positive cone and $<_P$ is Archimedean, we also say that P is Archimedean.

Examples 3.7.

- The usual orders on $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are Archimedean.
- Both $\mathbb{Z}^n, \mathbb{Q}^n$ can be granted Archimedean orders. We will construct orders for $\mathbb{Z}^2$ in the next section, in a way that is generalizable to $\mathbb{Z}^n$. For $\mathbb{Q}^n$, it suffices to note that any orders on $\mathbb{Z}^n$ can be extended to an order on $\mathbb{Q}^n$.
- The group $G = \langle x, y \,|\, y^{-1}xy = x^{-1}\rangle$ cannot be given an Archimedean order. Suppose that $<$ is an order on G. Up to isomorphism, we can assume that x, y are both positive. Given some n such that $x^n > y$, the following calculation shows a contradiction:

$$\begin{aligned} x^n > y &\implies yx^{-n}y^{-1} > y \\ &\implies x^{-n}y^{-1} > \mathrm{id} \\ &\implies y^{-1} > x^n > y > \mathrm{id} \quad \#. \end{aligned}$$

Definition 3.8. We denote $\mathrm{Ar}(G) \subset \mathrm{LO}(G)$ for the space of Archimedean positive cones of G.

Proposition 3.9. *The space of Archimedean orders* $\mathrm{Ar}(G)$ *is a* G_δ *subset of* $\mathrm{LO}(G)$*, thus is Polish with the inherited topology from* 2^G*.*

Proof. Consider the subbasic clopen sets of $\mathrm{LO}(G)$, which are of the form

$$U_g := \{P \in \mathrm{LO}(G) : g \in P\},$$

for $g \in G$.
Rewriting the Archimedean condition in term of positive cones, we get

$$\begin{aligned}\forall g, h \in P, \exists n \in \mathbb{N}, h^{-1}g^n \in P &\iff P \in \bigcap_{g,h\in P} \bigcup_{n\in\mathbb{N}} U_{h^{-1}g^n} \\ &\implies \mathrm{Ar}(G) = \bigcap_{g,h\in G} U_g \cap U_h \cap \left(\bigcup_{n\in\mathbb{N}} U_{h^{-1}g^n}\right).\end{aligned}$$

This is a countable intersection of open sets, hence a G_δ. ■

In general, if $\phi \in \mathrm{Aut}(G)$ and $P \in \mathrm{LO}(G)$, applying $\phi \cdot P := \{\phi(h) : h \in P\}$ gives us another positive cone. This gives us an action $\mathrm{Aut}(G) \curvearrowright \mathrm{LO}(G)$, which restricts to an action $\mathrm{Aut}(G) \curvearrowright \mathrm{Ar}(G)$. Furthermore, every automorphism acts as a homeomorphism of $\mathrm{LO}(G)$ and $\mathrm{Ar}(G)$.

In the case when G is finitely generated, $\mathrm{Aut}(G)$ is countable, meaning that $E_{\mathrm{LO}(G)}^{\mathrm{Aut}(G)}$ is a countable Borel equivalence relation.

3.1 Description of Archimedean orders on $\mathbb{Z}^2$

Archimedean orders on $\mathbb{Z}^2$ can be characterized precisely using the following observation.

Proposition 3.10. *Every positive cone* $P \in \mathrm{LO}(\mathbb{Z}^2)$ *contains a set of the form* $H_P \cap \mathbb{Z}^2$*, where* H_P *is a unique open half-plane with* $(0,0) \in \partial H_P$*.*

Note that we will stop mentioning that half-planes are open and just say half-planes.

Proof. Start by extending P to a positive cone $P^* \in \mathrm{LO}(\mathbb{Q}^2)$ as follows:

$$P^* := \mathbb{Q}_{>0} \cdot P = \left\{v \in \mathbb{Q}^2 : \exists n \in \mathbb{N}_{>0},\, n \cdot v \in P\right\}.$$

Then, consider a partition of $\mathbb{R}^2$ into three pieces:

$$\begin{aligned}R_1 =& \{x \in \mathbb{R}^2 : \exists\varepsilon,\, \forall x' \in B(x,\varepsilon) \cap \mathbb{Q}^2,\, x' \in P^*\},\\ R_2 =& \{x \in \mathbb{R}^2 : \exists\varepsilon,\, \forall x' \in B(x,\varepsilon) \cap \mathbb{Q}^2,\, x' \in -P^*\},\\ R_3 =& \{x \in \mathbb{R}^2 : \forall\varepsilon,\, \exists x_+, x_- \in B(x,\varepsilon) \cap \mathbb{Q}^2,\, x_+ \in P^* \text{ and } x_- \in -P^*\}.\end{aligned}$$

There are two main observations that allows us to establish the proposition:

- Both R_1 and R_2 are non-empty open sets.
- R_3 is a non-empty vector subspace of $\mathbb{R}^2$.

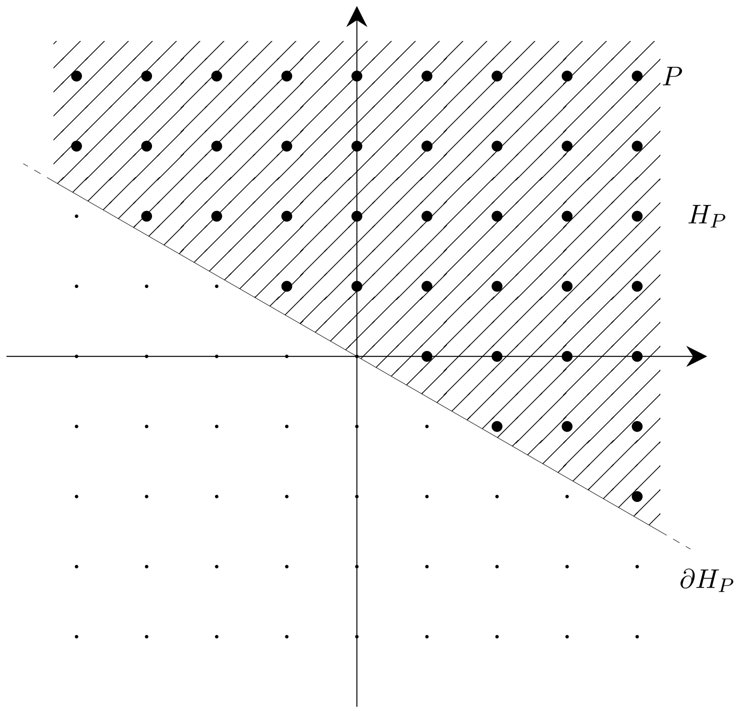

Figure 1: A positive cone in $\mathbb{Z}^2$. Bigger dots represent elements of P and the shaded region is H_P.

To establish the first observation, note that $R_2 = -R_1$. As such, we deal only with R_1. Openness is direct from the definition: if ε witnesses that $x \in R_1$, $B(x, \varepsilon) \subset R_1$.

Out of $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}-1\\0\end{pmatrix}$, exactly one is positive. Similarly for the pair $\begin{pmatrix}0\\1\end{pmatrix}$ and $\begin{pmatrix}0\\-1\end{pmatrix}$. Since P^* is closed under addition, at least one quadrant must be entirely positive and so the reals contained in its interior are in R_1.

As for the second observation, given $x, y \in R_3$ and ε, one can pick two vectors $\tilde{x} \in B(x, \frac{\varepsilon}{2}) \cap \mathbb{Q}^2$ and $\tilde{y} \in B(y, \frac{\varepsilon}{2}) \cap \mathbb{Q}^2$ such that $\tilde{x}, \tilde{y} \in P^*$. Since positive cones are closed under addition, $\tilde{x} + \tilde{y} \in P^*$ and $\tilde{x} + \tilde{y} \in B(x + y, \varepsilon)$. We can do a similar argument to find a vector in $-P^* \cap B(x + y, \varepsilon)$.

Scalar multiplication is done exactly in the same manner. Since there are both positive and negative vectors, there are arbitrarily small positive and negative vectors, getting that $(0, 0) \in R_3$. Thus, R_3 is a vector subspace of $\mathbb{R}^2$.

However, since $\mathbb{R} - R_3 = R_1 \sqcup R_2$ is a union of two non-empty open sets, it must be a non-empty disconnected set. The only subspace that can disconnect $\mathbb{R}^2$ in this fashion is a line, so that R_3 is a line through the origin, with R_1, R_2 being the half-planes on either side of it.

It can then be verified that $P \supset R_1 \cap \mathbb{Z}^2$ and that $(0,0) \in R_3 = \partial R_1$. We take $H_P := R_1$. Uniqueness is achieved by the fact that if we are given a different half-plane H with $\partial H \ni (0,0)$, it must intersect $R_2 \cap \mathbb{Z}^2$ non-trivially, meaning it is not contained in P. ■

Remark 3.11. We say that the line ∂H_P **delimits** P, where H_P comes from Proposition 3.10. Note that by that proposition, for every cone $P \in \mathrm{LO}(\mathbb{Z}^2)$, there is exactly one line Δ which delimits it.

Remark 3.12. We will not need the full strength of this characterization, but Proposition 3.10 allows us to describe the left-orders and the Archimedean orders of $\mathbb{Z}^n$ for any n.

We denote by Irr the set of irrational numbers. We denote by $\mathbb{P}^1_{\mathrm{Irr}}$ the set of lines $\Delta \subset \mathbb{R}^2$ such that $\Delta \cap \mathbb{Z}^2 = \{(0,0)\}$.

Definition 3.13. We define $\mu : \mathbb{P}^1_{\mathrm{Irr}} \to \mathrm{Irr}$ such that $\mu(\Delta)$ is the unique irrational such that

$$\begin{pmatrix} \mu(\Delta) \\ 1 \end{pmatrix} \in \Delta.$$

This can be shown to be a bijection.

Proposition 3.14. *A line $\Delta \subset \mathbb{R}^2$ is in $\mathbb{P}^1_{\mathrm{Irr}}$ if and only if it delimits an Archimedean positive cone.*

Proof. Let $\Delta \subset \mathbb{R}^2$ be a line and $P \in \mathrm{LO}(\mathbb{Z}^2)$ a positive cone delimited by it. Pick a vector $v \perp \Delta$ and consider the functional $f_v(x) = \langle v, x \rangle$, where $\langle \cdot, \cdot \rangle$ corresponds to the usual scalar product.

We can assume that P, v are on the same side of Δ, meaning that for every vector $x \in P, f_v(x) \geq 0$. Similarly, $x \in P^c \Rightarrow f_v(x) \leq 0$.

Suppose that P is not Archimedean. We want to show that $P \cap \Delta$ is non-empty. Since P is not Archimedean, there are $x, y \in P$ with $\forall n,\ nx < y$. Using linearity of f_v:

$$\begin{aligned} & \forall n,\ nx < y \\ \iff & \forall n,\ 0 < y - nx \\ \iff & \forall n,\ 0 \leq f_v(y - nx) \\ \iff & \forall n,\ 0 \leq f_v(y) - n f_v(x) \end{aligned}$$

This is only possible if $f_v(x) = 0$, meaning that $x \in \Delta$, as required.

For the other implication, suppose that there is some vector $x \in P \cap \Delta$. Let $y \in P - \Delta$. Again using linearity and that fact that $f_v(x) > 0$ implies that $x \in P$,

$$\begin{aligned} & \forall n,\ 0 < f_v(y) - n f_v(x) \\ \Longrightarrow & \forall n, y - nx \in P \\ \iff & \forall n, nx < y. \end{aligned}$$

Hence, P is not Archimedean, concluding the proof. ■

This allows us to define the following map:

Definition 3.15. We define $\Lambda : \mathrm{Ar}(\mathbb{Z}^2) \to \mathbb{P}^1_{\mathrm{Irr}}$ as the map associating to a positive cone P the line $\Lambda(P) = \partial H_P$, which delimits it. This is well-defined in light of Proposition 3.10, Remark 3.11 and Proposition 3.14.

Remark 3.16. When $\mathbb{P}^1_{\mathrm{Irr}}$ is given the topology induced by μ, Λ becomes continuous, hence Borel.

Proposition 3.17. *The map $\Lambda : \mathrm{Ar}(\mathbb{Z}^2) \to \mathbb{P}^1_{\mathrm{Irr}}$ is surjective and 2-1. Furthermore, every pre-image is of the form $\{P, -P\}$.*

Proof. For surjectivity, note that by the forward implication of Proposition 3.14, every line $\Delta \in \mathbb{P}^1_{\mathrm{Irr}}$ delimits at least one Archimedean positive cone.

For 2-1, consider $\Delta \in \mathbb{P}^1_{\mathrm{Irr}}$ and the two open half-plane on either side of it, H_1, H_{-1}. Given $P \in \Lambda^{-1}(\Delta)$ with $P \supset H_P \cap \mathbb{Z}^2$, we know H_P is equal to either H_1 or H_{-1}, since these are the only two half-planes with boundary Δ.
Since $\Delta \cap \mathbb{Z}^2 = \{(0,0)\}$ and $H_{-1} = -H_1$, we have that

$$\begin{aligned}
\left(H_i \cap \mathbb{Z}^2\right) \subset P &\Longrightarrow (H_{-i} \cap \mathbb{Z}^2) \subset -P \\
&\Longrightarrow H_{-i} \cap P = \emptyset \\
&\Longrightarrow P \subset (H_i \cap \mathbb{Z}^2) \cup (\Delta \cap \mathbb{Z}^2) \\
&\Longrightarrow P = H_i \cap \mathbb{Z}^2.
\end{aligned}$$

Thus there are exactly two cones in the pre-image of Δ:

$$\Lambda^{-1}(\Delta) = \left\{H_1 \cap \mathbb{Z}^2, H_{-1} \cap \mathbb{Z}^2\right\}$$

These cones are opposites, as required. ■

Remark 3.18. This proof tells us a little bit more, that in the Archimedean case, Proposition 3.10 is refined so that $P = H_P \cap \mathbb{Z}^2$.

4 Borel complexity of Archimedan orders

In this section, we wrap up the work of the previous sections and discuss Archimedean orders of $\mathbb{Z}^2$ in the context of Borel complexity and present our result, Theorem 1.5.

Remember that since $\mathbb{P}^1_{\mathrm{Irr}}$ is composed of lines $\Delta \subset \mathbb{R}^2$ such that $\Delta \cap \mathbb{Z}^2 = \{(0,0)\}$, $\mathrm{GL}_2(\mathbb{Z})$ acts on it through its action on $\mathbb{R}^2$, which fixes both $\mathbb{Z}^2$ and the origin. We now wish to relate the isomorphism relations on $\mathrm{Ar}(\mathbb{Z}^2)$ with this action on $\mathbb{P}^1_{\mathrm{Irr}}$, which we can do using the two following propositions.

Proposition 4.1. *The map $\Lambda : \mathrm{Ar}(\mathbb{Z}^2) \to \mathbb{P}^1_{\mathrm{Irr}}$ is a $\mathrm{GL}_2(\mathbb{Z})$-invariant reduction from $E^{\mathrm{Ar}(\mathbb{Z}^2)}_{\mathrm{GL}_2(\mathbb{Z})}$ to $E^{\mathbb{P}^1_{\mathrm{Irr}}}_{\mathrm{GL}_2(\mathbb{Z})}$.*

Proof. As noted in the remark following Proposition 3.17, in the Archimedean case, $P = H_P \cap \mathbb{Z}^2$. We get that when $M \in \mathrm{GL}_2(\mathbb{Z})$,

$$\begin{aligned}\Lambda(M \cdot P) &= \Lambda\left(M \cdot \left(H_P \cap \mathbb{Z}^2\right)\right)\\ &= \Lambda\left((M \cdot H_P) \cap \mathbb{Z}^2\right).\end{aligned}$$

Since $\mathrm{GL}_2(\mathbb{Z})$ acts continuously,

$$\begin{aligned}\partial\left(M \cdot H_P\right) &= M \cdot \partial H_P\\ \implies \Lambda(M \cdot P) &= M \cdot \Lambda(P).\end{aligned}$$

Due to Proposition 2.13, the only thing left to show is that pre-images of Λ are themselves $\mathrm{GL}_2(\mathbb{Z})$-invariant. Since pre-images are all of the form $\{P, -P\}$ by Proposition 3.17 and that

$$-P = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \cdot P,$$

we get that Λ is a reduction, as required. ■

Proposition 4.2. *There is a continuous right-inverse $\iota : \mathbb{P}^1_{\mathrm{Irr}} \to \mathrm{Ar}(\mathbb{Z}^2)$ of Λ which is a reduction from $E^{\mathbb{P}^1_{\mathrm{Irr}}}_{\mathrm{GL}_2(\mathbb{Z})}$ to $E^{\mathrm{Ar}(\mathbb{Z}^2)}_{\mathrm{GL}_2(\mathbb{Z})}$.*

Proof. Given $\Delta \in \mathbb{P}^1_{\mathrm{Irr}}$, there are two choices for $\iota(\Delta)$, since for some $P \in \mathrm{Ar}(\mathbb{Z}^2)$,

$$\Lambda^{-1}(\Delta) = \{P, -P\}.$$

Out of these cones, exactly one contains the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Take this positive cone for $\iota(\Delta)$. Since Λ is $\mathrm{GL}_2(\mathbb{Z})$-invariant, by Proposition 4.1 we get

$$\begin{aligned}\iota(M \cdot \Delta) &\in \Lambda^{-1}(M \cdot \Delta)\\ &= M \cdot \{\iota(\Delta), -\iota(\Delta)\}\\ &= \{M \cdot \iota(\Delta), -M \cdot \iota(\Delta)\}\end{aligned}$$

Thus, we either get

$$\iota(M \cdot \Delta) = M \cdot \iota(\Delta)$$
$$\text{or } \iota(M \cdot \Delta) = -M \cdot \iota(\Delta).$$

In both cases, we get

$$\iota(\Delta) \; E^{\mathrm{GL}_2(\mathbb{Z})}_{\mathrm{Ar}(\mathbb{Z}^2)} \; \iota(M \cdot \Delta).$$

This proves that

$$\Delta \; E^{\mathrm{GL}_2(\mathbb{Z})}_{\mathbb{P}_1^{\mathrm{Irr}}} \; \Delta' \Longrightarrow \iota(\Delta) \; E^{\mathrm{GL}_2(\mathbb{Z})}_{\mathrm{Ar}(\mathbb{Z}^2)} \; \iota(\Delta').$$

The reverse implication is given by the fact that Λ is a reduction and $\Lambda \circ \iota = \mathrm{id}$. ■

We wish to reduce the orbit equivalence relation $E^{\mathbb{P}^1_{\mathrm{Irr}}}_{\mathrm{GL}_2(\mathbb{Z})}$ to an equivalence relation on Irr. In our case, this is the orbit equivalence relation of the action by Möbius transformation, defined as follows:

Definition 4.3. We define the action by **Möbius transformations** $\mathrm{GL}_2(\mathbb{Z}) \curvearrowright \mathrm{Irr}$ to be

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \alpha := \frac{a\alpha + b}{c\alpha + d}$$

Proposition 4.4. *The Borel equivalence relations* $E^{\mathbb{P}^1_{\mathrm{Irr}}}_{\mathrm{GL}_2(\mathbb{Z})}$ *and* $E^{\mathrm{Irr}}_{\mathrm{GL}_2(\mathbb{Z})}$ *are bireducible.*

Proof. We claim that the map $\mu : \mathbb{P}^1_{\mathrm{Irr}} \to \mathrm{Irr}$ is a reduction from $E^{\mathbb{P}^1_{\mathrm{Irr}}}_{\mathrm{GL}_2(\mathbb{Z})}$ to $E^{\mathrm{Irr}}_{\mathrm{GL}_2(\mathbb{Z})}$ and its inverse is a reduction from $E^{\mathrm{Irr}}_{\mathrm{GL}_2(\mathbb{Z})}$ to $E^{\mathbb{P}^1_{\mathrm{Irr}}}_{\mathrm{GL}_2(\mathbb{Z})}$.

Since μ is a bijection, it is sufficient to prove that it is $\mathrm{GL}_2(\mathbb{Z})$-invariant, applying Proposition 2.13.

Let $\Delta \in \mathbb{P}^1_{\mathrm{Irr}}$ and define $\alpha_\Delta := \mu(\Delta)$. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$.

$$\begin{aligned}
&\begin{pmatrix} \alpha_\Delta \\ 1 \end{pmatrix} \in \Delta, \\
\Longrightarrow\, & M \cdot \begin{pmatrix} \alpha_\Delta \\ 1 \end{pmatrix} \in M \cdot \Delta \\
\Longrightarrow\, & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \alpha_\Delta \\ 1 \end{pmatrix} \in M \cdot \Delta \\
\Longrightarrow\, & \begin{pmatrix} a\alpha_\Delta + b \\ c\alpha_\Delta + d \end{pmatrix} \in M \cdot \Delta \\
\Longrightarrow\, & \frac{1}{c\alpha_\Delta + d} \cdot \begin{pmatrix} a\alpha_\Delta + b \\ c\alpha_\Delta + d \end{pmatrix} \in M \cdot \Delta \\
\Longrightarrow\, & \begin{pmatrix} M \cdot \alpha_\Delta \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{a\alpha_\Delta + b}{c\alpha_\Delta + d} \\ 1 \end{pmatrix} \in M \cdot \Delta \\
\Longrightarrow\, & \mu\left(M \cdot \Delta\right) = M \cdot \mu(\Delta),
\end{aligned}$$

as required. ■

Theorem 4.5. *The action* $\mathrm{GL}_2(\mathbb{Z}) \curvearrowright \mathrm{Ar}(\mathbb{Z}^2)$ *is bireducible with* $\mathrm{GL}_2(\mathbb{Z}) \curvearrowright \mathrm{Irr}$.

Proof. From Proposition 4.4, both μ, μ^{-1} are reductions. From Propositions 4.1, 4.2, Λ, ι are reductions. Using Proposition 2.13, a composition of reductions is still a reduction. Thus, the two following functions are reduction:

$$\mu \circ \Lambda : \mathrm{Ar}(\mathbb{Z}^2) \to \mathrm{Irr}$$
$$\iota \circ \mu^{-1} : \mathrm{Irr} \to \mathrm{Ar}(\mathbb{Z}^2).$$

■

As for the complexity of $E_{P^1_{\mathrm{Irr}}}^{\mathrm{GL}_2(\mathbb{Z})}$, we cite the following theorem:

Theorem 4.6 ([7])**.** *The action by Möbius transformations* $\mathrm{GL}_2(\mathbb{Z}) \curvearrowright \mathrm{Irr}$ *is hyperfinite, but not smooth.*

Sketch of proof. The proof rests on a crucial fact proved in [5, thm 175], reformulated in [7] in the context of Borel equivalence relations. The crucial fact states, with our notation, that if $\alpha, \beta \in \mathrm{Irr}$,

$$\exists M \in \mathrm{GL}_2(\mathbb{Z}), \alpha = M \cdot \beta \iff \overline{\alpha}\ E_t\ \overline{\beta},$$

where $\overline{x}$ denotes the continued fraction of x, that is the sequence (a_i) with

$$x = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}.$$

The equivalence relation E_t on $\mathbb{N}^{\mathbb{N}}$ is the tail equivalence relation, defined as follows:

$$\overline{a}\, E_t\, \overline{b} \iff \exists i, j,\, \forall n, a_{i+n} = b_{j+n}.$$

E_t is known to be hyperfinite, but not smooth, as cited in [7]. ■

Joining Theorems 4.5 and 4.6, we get a proof of Theorem 1.5:

Proof of Theorem 1.5. We wish to show that the isomorphism relation on $\mathrm{Ar}(\mathbb{Z}^2)$ is hyperfinite, but not smooth. From Theorem 4.5 we get that $E_{\mathrm{Ar}(\mathbb{Z}^2)}^{\mathrm{GL}_2(\mathbb{Z})}$ is bireducible to $E_{P^1_{\mathrm{Irr}}}^{\mathrm{GL}_2(\mathbb{Z})}$. By Theorem 4.6, this is hyperfinite, but not smooth. ■

5 Conclusion

There are multiple interesting ways to expand on this line of research. As mentioned, a question of [2] is whether Theorem 1.3 can be strengthened to show hyperfiniteness. We ask the following intermediate question:

Question 5.1. *Let $n \in \mathbb{Z}$ and $\mathbb{Z}[\frac{1}{n}]$ be the associated ring extension of $\mathbb{Z}$. Is the isomorphism relation on Archimedean ordering of $\mathbb{Z}[\frac{1}{n}]^2$ hyperfinite, but not smooth?*

The isomorphism relation for Archimedean orders of $\mathbb{Q}^2$ is an increasing union of the isomorphism relations for $\mathbb{Z}[\frac{1}{n!}]$. A positive answer to this question would thus give an interesting example to study the conjecture stating that an increasing union of hyperfinite relations is itself hyperfinite.

There is also a possible generalization to higher dimensions, as some tools we use to characterize Archimedean orderings generalize nicely using the Grassmannian $\mathrm{Gr}(n-1, \mathbb{R}^n)$, that is the space of hyperplanes of dimension $n-1$ in $\mathbb{R}^n$.

Question 5.2. *Is the isomorphism relation on the Archimedean orderings of $\mathbb{Z}^n$ hyperfinite but not smooth?*

Question 5.2 is of particuliar interest. By classical results, Archimedean-ordered groups are abelian and left-orderable groups have no torsion. Thus, every finitely generated Archimedean-orderable groups if of the form $\mathbb{Z}^n$ for some n, by the structure theorem mentionned in 1.1. This makes the isomorphism relation on finitely generated Archimedean ordered groups highly dependent on the relations for each $\mathbb{Z}^n$. As an example, a positive answer to Question 2 would ensure that isomorphism of Archimedededean-ordered finitely generated groups is hyperfinite.

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